College of Engineering

Hydraulic Structures & Water Resources Department

Mathematics Lectures

for

First Stage

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References :

1. *Calculus By Thomas*
2. *Schaum Series*

Main Objects

First Semester :

1. Coordinates & Graphs in the Plane
2. Slopes & Equations for Lines
3. Functions & their Graphs
4. Conic Sections
5. Limits
6. Continuity
7. Differentiation & Applications

Second Semester :

8. Integration
9. Application on Definite Integral
10. Transcendental Functions
11. Methods of Integration

Geometrical Identities

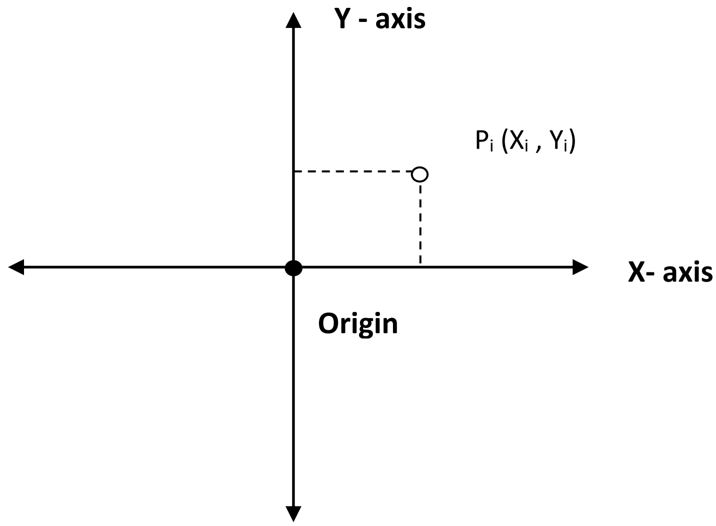
1. $\sin(\theta+2\pi) = \sin\theta$
2. $\cos(\theta+2\pi) = \cos\theta$
3. $\sin^2(\theta) + \cos^2\theta = 1$
4. $\sin(-\theta) = -\sin\theta$
5. $\cos(-\theta) = \cos\theta$
6. $\sin(x+y) = \sin x\cos y + \cos x\sin y$
7. $\cos(x+y) = \cos x\cos y - \sin x\sin y$
8. $\sin(x-y) = \sin x\cos y - \cos x\sin y$
9. $\cos(x-y) = \cos x\cos y + \sin x\sin y$
10. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
11. $\sin 2x = 2\sin x\cos x$
12. $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
13. $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
14. $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
15. $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$
16. $1 + \tan^2 x = \sec^2 x$
17. $1 + \cot^2 x = \csc^2 x$
18. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
19. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
20. $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$
21. $\sec x = 1/\cos x$
22. $\csc x = 1/\sin x$

Chapter One

COORDINATES & GRAPHS IN THE PLANE

1.1. *Introduction* :

To assign coordinates to points in a plane, we start with two number lines that cross at their zero points at right angles. Each line represents the real numbers that can be represented by decimals. Figure below shows the usual way of drawing of lines, with one line horizontal called the **x – axis** and the other vertical line is called the **y – axis**. The point at which the lines cross is the **origin**.



1.2. *Distance between Points* :

Distance formula between two points P(x_1 , y_1) & Q(x_2 , y_2) is :

ΔX = (X of terminal point) – (X of initial point) = $X_2 - X_1$

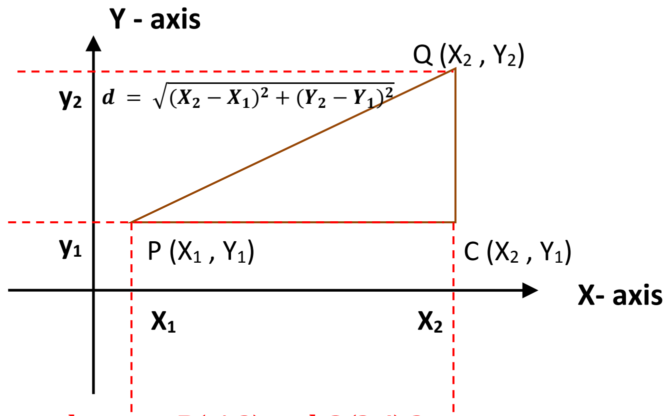
$\Delta Y = Y_2 - Y_1$

If P_1 (X_1,Y_1) , P_2 (X_2,Y_2) , the distance (d) between two points is :

$$d = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

$$\boldsymbol{d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}$$

This formula is depending on the Pythagorean Theorem to triangle as shown in the fig. below :



Example (1) : find the distance between P(-1,2) and O(3,4) ?

Solution :

d = $\sqrt{(3-(-1))^2+(4-2)^2} = 2\sqrt{5}$

H.W(2): Find the distance between $P_1(-1,2)$, $P_2(2,-2)$?

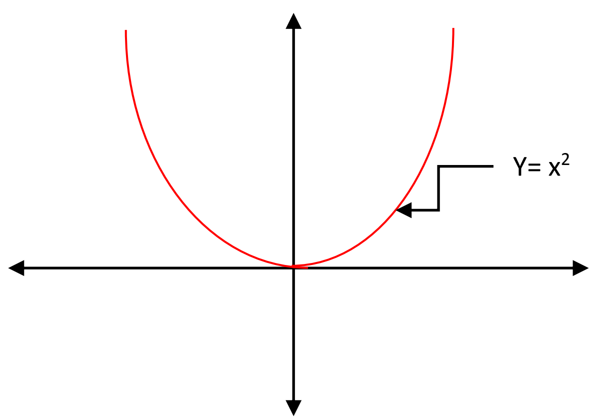
1.3. *Graphs of Equations* :

The points (x,y) whose coordinates satisfy an equation in the xy – plane. Graphs give us a practical way to picture equations as lines or curves.

Example (2) : Graph the equation $y = x^2$ for the values of x from x = -2 to x = 2 ?

Solution :

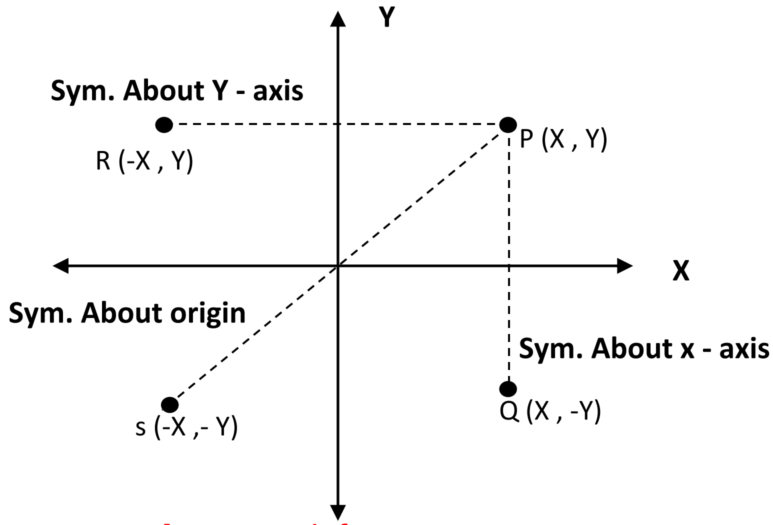
x	2	1	0	-1	-2
y	4	1	0	1	4



1.4. Symmetry :

We can use coordinates formulas to describe important symmetries in the coordinate plane as shown in the fig. below :

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Example (3) : show the symmetry about any axis for :

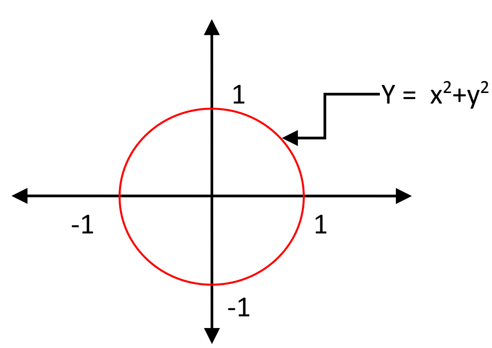
1. **P(5,2) & Q(5,-2)**
2. **P(5,2) & Q(-5,-2)**
3. **P(5,2) & Q(-5,-2)**

Solution :

1. Sym. about x-axis 2. Sym. about y-axis 3. Sym. about origin

Example (4) : The graph of $x^2 + y^2 = 1$ has all of three symmetries listed in the fig. below :

Solution :



1. Sym. about x – axis :
   $x^2 + (-y)^2 = 1 \Longrightarrow x^2 + y^2 = 1 \quad \therefore$ sym. about x-axis
2. Sym. about y – axis :
   $(-x)^2 + y^2 = 1 \Longrightarrow x^2 + y^2 = 1 \quad \therefore$ sym. about y-axis
3. Sym. about origin :
   $(-x)^2 + (-y)^2 = 1 \Longrightarrow x^2 + y^2 = 1 \quad \therefore$ sym. about origin

Note :

- The function is called “Even” for all x symmetry about y – axis.
- The function is called “Odd” for all x symmetry about x – axis.

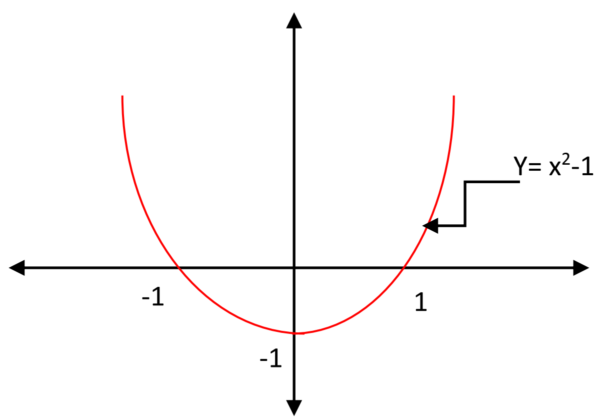
1.5. *Intercepts :*

The easiest points to find on an equation's graph are the points where the graph touches or crosses the axes. These points (when they exists) are the graphs intercepts, we find the x- intercepts by setting y equal to zero and y- intercepts by setting x equal to zero.

Example (5) : Find the intercepts of the graph of the equation $y = x^2 - 1$?

Solution :

1. x intercept :
   $x^2 - 1 = 0 \Longrightarrow x^2 = 1 \Longrightarrow (x = \pm 1 \quad , \quad y = 0)$
2. y intercept :
   $y = -1$ at $x = 0$



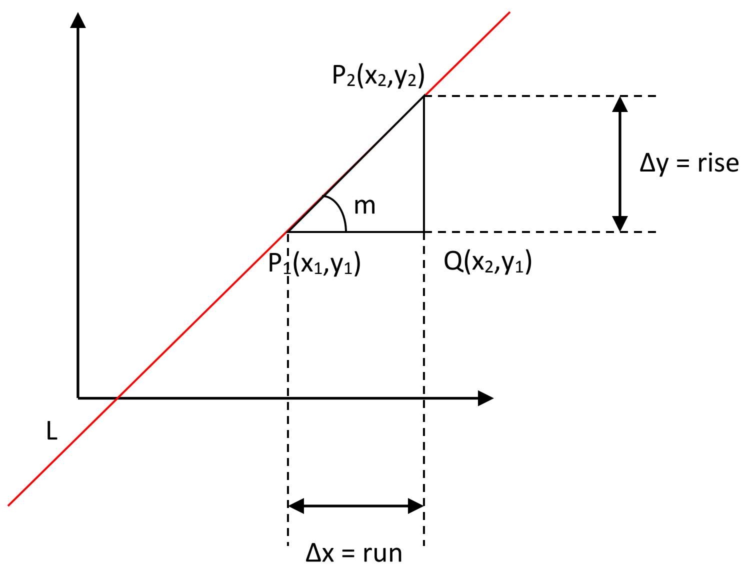
Chapter Two

SLOPE AND EQUATIONS FOR LINES

2.1. Increments :

When a particle moves from $P_1(x_1 , y_1)$ to $P_2(x_2 , y_2)$, the increments are :

$\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$



Example (1) : Find the net change in coordinate between :

1. **A(4,-3) , B(2,5)**
2. **C(5,6) , D(5,1)**

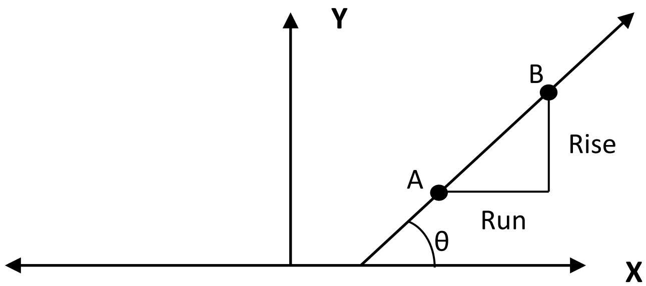
Solution :

1. $\Delta x = 2 - 4 = -2$
   $\Delta y = 5-(-3) = 8$
2. $\Delta x = 5 - 5 = 0$
   $\Delta y = 1-6 = -5$

2.2. Slopes of Non – Vertical Line :

1. Slope of vertical line = tan 90° = ∞
2. Slope of horizontal line = tan 0° = 0
3. All lines except vertical line have slopes. We calculate slopes from change in coordinates, then the slopes of non – vertical line is :

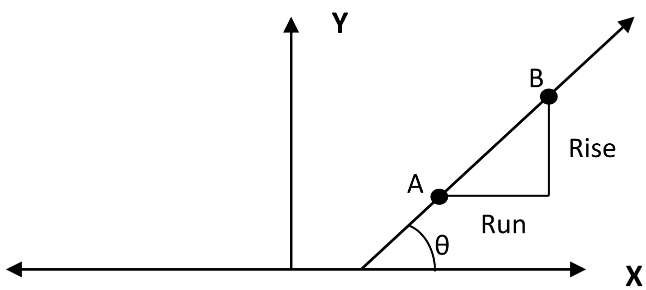
Slope (m) = Rise / Run = $\frac{\Delta Y}{\Delta X} = \frac{(Y_2 - Y_1)}{(X_2 - X_1)}$ = tan θ



Notes :

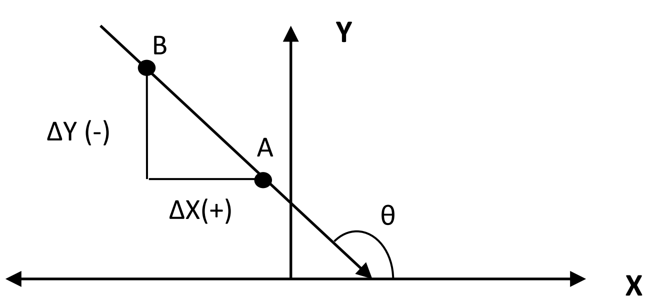
A. For a line have a slope upward to the right , ΔY is positive when ΔX is also :

$0 < \theta < 90^0$ and tan θ = $\frac{\Delta Y}{\Delta X}$ is positive

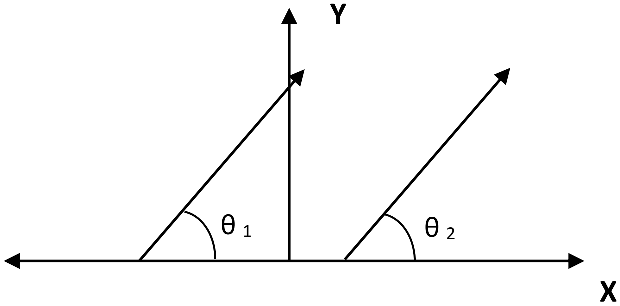


B . For a line have a slope downward to the right , ΔY is negative when ΔX is positive :

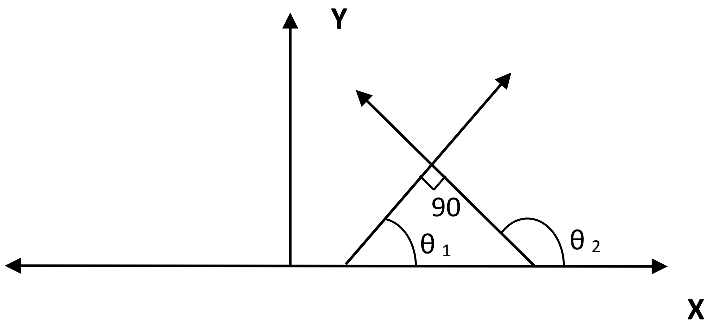
$90^0 < \theta < 180^0$ and tan θ = $\frac{\Delta Y}{\Delta X}$ is negative



C. For a horizontal line , tan θ = 0
D. For a vertical lines, tan θ = does not exist
E. Two parallel lines have equal angles, hence equal slopes , $m_1 = m_2$



F. Two perpendicular lines ,which is :
$\theta_2 = 90 + \theta_1$
$\tan\theta_2 = \tan(90 + \theta_1) = -\cot\theta_1$
$\tan\theta_2 = -\frac{1}{\tan\theta_1} \Longrightarrow m_2 = -\frac{1}{m_1} \Longrightarrow m_1 * m_2 = -1$



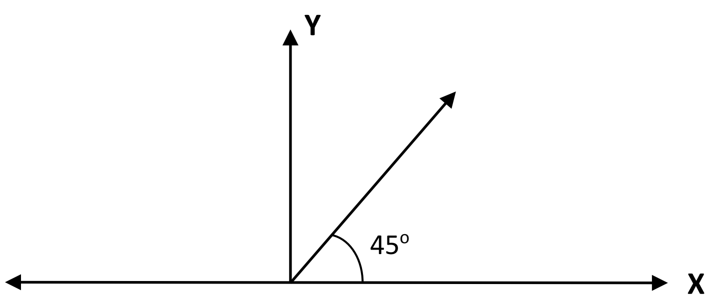
Note 1 : To find the angle between two lines there slopes are known :

$\theta_2 = \theta + \theta_1 \Longrightarrow \theta = \theta_2 - \theta_1 \Longrightarrow \tan\theta = \tan(\theta_2 - \theta_1)$

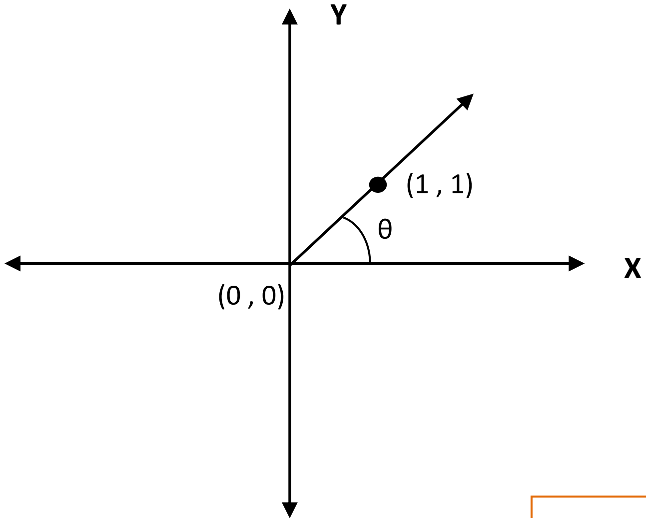
$$\tan\theta = \frac{\tan\theta 2 - \tan\theta 1}{1 + \tan\theta 2 * \tan\theta 1} = \frac{m2 - m1}{1 + m2 * m1}$$

Note 2: If tan θ = 45^0 , you can find any point lies on the line passes through the origin :

$m = \tan\theta = \tan 45 = 1 \Longrightarrow 1 = \frac{y2 - y1}{x2 - x1} = \frac{0 - y1}{0 - x1} \Longrightarrow y_1 = x_1 = 1$



Example (2) : A straight line is draw through the point (0,0) and the point (1,1) , what acute angle does it make with the positive x – axis ? sketch.



Solution :

$\tan\theta = y/x = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1) = 1/1 = 1 \Longrightarrow$ $\boxed{\theta = 45^0}$

Example (3) : Are the three points $P_1(-1,5)$, $P_2(1,3)$ and $P_3(7,12)$ lies on a straight line ?

Solution :

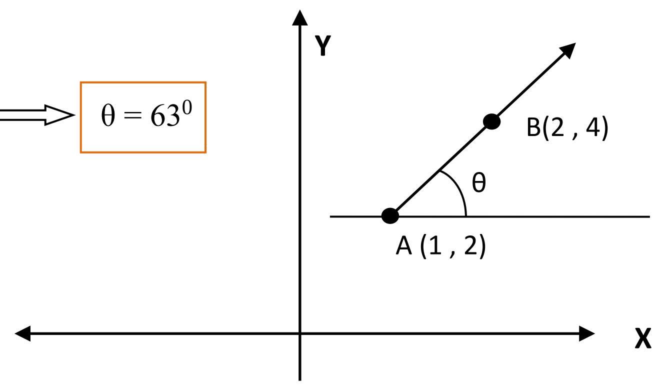
$m_1 = \Delta y / \Delta x = (5-3)/(-1-1) = 2/-2 = -1$, $m_2 = (12-3)/(7-1) = 9/6 = 1.5$

$m_1 \neq m_2$ therefore ; the points don't lie on a straight line.

Example (4) : A straight line is draw through the point A(1,2) and the point B(2,4) , what acute angle does it make with the horizontal line through point A ? sketch.

Solution :

$\tan\theta = y/x = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1) = 2/1 = 2 \Longrightarrow$ $\boxed{\theta = 63^0}$



H.W: The line through the pair of points (2,3) and (1,1) cuts the y-axis at the point (0,b), Find the value of (b) by using similar triangles ? sketch.

2.3. *Point – Slope Equation* :

The point – slope equation of the line that passes through the point (x_1 , y_1) with slope (m) is :

$$m = \frac{y - y_1}{x - x_1}$$

Example (5) : Write an equation of the line that passes through the point (2,3) with slope (-2/3) ?

Solution :

-2/3 = (y-3) / (x-2) ⟹ y = (-3/2) x + 6

Example (6) : Write an equation of the line that passes through the points (2,3) & (3,4) ?

Solution :

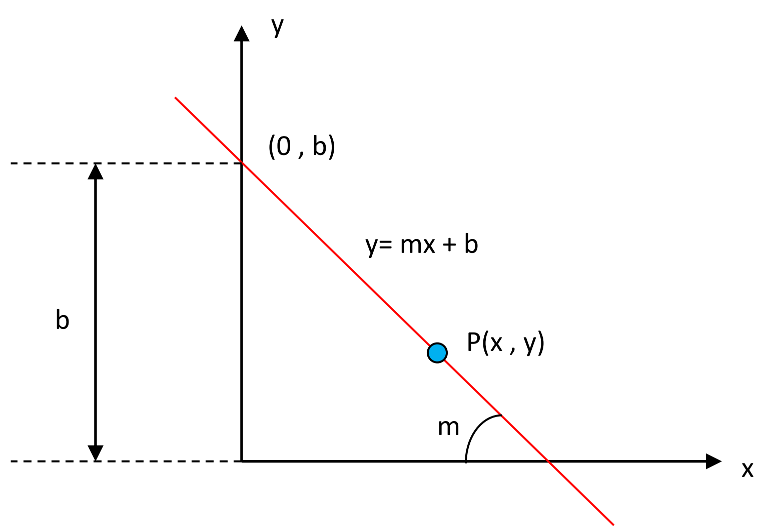
m = (4-(-1)) / (3-(-2)) = 5/5 = 1

1 = (y-4) / (x-3) ⟹ y = x + 1

2.4. *Slope - Intercept Equation* :

This equation is derived & based by knowing slope and y – intercept (b) :

$$y = mx + b$$



Example (7) : Find the slope of the line 8x + 5y = 20 ?

Solution :

$y = (-8/5)\,x + 4 \Longrightarrow$ $m = (-8/5)\ ,\ b = 4$

2.5. *The General Linear Equation :*

The general form of linear equation is :

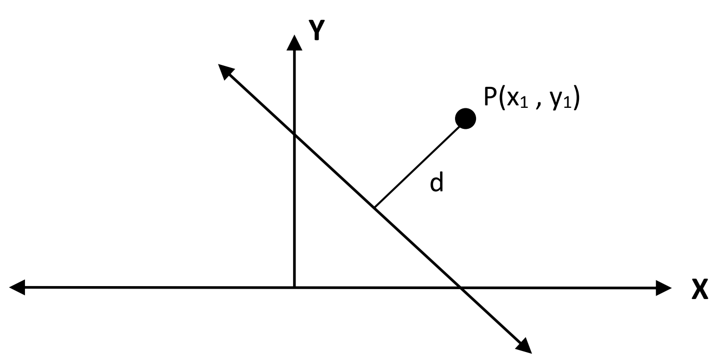
$$Ax + By + C = 0$$

2.6. *The Distance from a Point to a Line :*

To calculate the distance from a point $P(x_1,y_1)$ to a line L, we find the point $Q(x_2,y_2)$ at the end of line $\bar{L}$ that passes through P and cross vertically L at point Q and by knowing the coordinates of P and Q, we can find the distance between them.

We apply the following formula to find the distance (d) from point (x_1 , y_1) to straight line $Ax_1+By_1+C=0$:

$$d = \frac{|Ax1+By1+C|}{\sqrt{A^2+B^2}}$$



Example (8) : Find the distance from the point P(2,1) to the line y = x + 2 ?

Solution :

1. Find point Q depending on the slopes of L & $\bar{L}$
   $m_L = -1/m\bar{L}$, y = x+2 …. (1)
   m = 1 and $m\bar{L} = -1$, P (2,1)
   -1 = (y – 1) / (x – 2)
   y = -x + 3 …..(2)

2. Solving eq. (1) & (2) simultaneously : -x+3 = x+3 ⟹ x = ½ , y = 5/2

$$d = \sqrt{(2-\tfrac{1}{2})^2 + (1-\tfrac{5}{2})^2} = \boxed{\tfrac{3}{2}\sqrt{2}}$$

Example (9): Find the distance from the point (-3,5) to the line 2x + 3y = 5 ?

Solution:

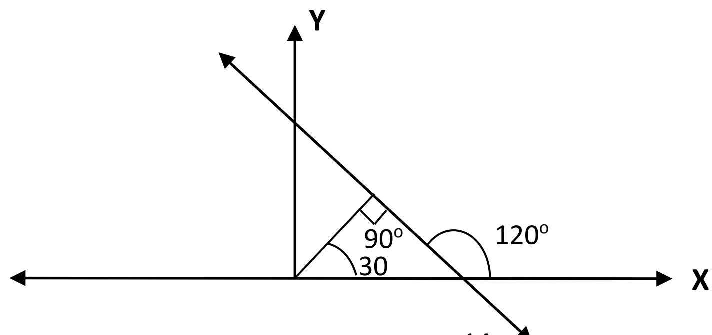
2x + 3y – 5 = 0 ⟹ A = 2 , B = 3 , C = -5

$x_1 = -3$, $y_1 = 5$

$$d = \frac{|2*-3+3*5-5|}{\sqrt{4+9}} = \boxed{\frac{4}{\sqrt{13}} \text{ units}}$$

H.W: Find the line that passes through the point (1,2) and it's parallel to the line x+2y = 3 ?

Example (10): Find the equation of the straight line at distance of 2 units from origin and making an angle of 120° with the positive direction of x - axis ?



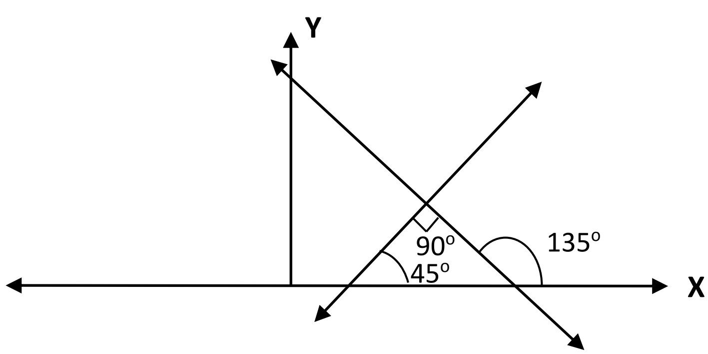
Solution :

$\cos 30^o = 2 / x \Longrightarrow \frac{\sqrt{3}}{2} = \frac{2}{x} \Longrightarrow x = \frac{4}{\sqrt{3}}$ & $y = 0$

$\tan 120^o = \tan (90+30) = -\cot 30^o = -1/\tan 30^o = \frac{-1}{\frac{1}{\sqrt{3}}} = -\sqrt{3}$

$y - y_1 = m (x - x_1) \Longrightarrow$ $\boxed{\mathbf{y = -\sqrt{3}\,X + 4}}$

Example (11): Find the equation of the two straight lines passing through (5,2) and cutting the x-axis at 45^o ?



Solution :

$m_1 = \tan 45^o = 1$; $m_2 = \tan 135^o = \tan (90+45) = -\cot 45^o = -1/\tan 45^o = \frac{-1}{1} = -1$

$y - y_1 = m (x-x_1) \Longrightarrow y - 2 = -1 (x - 5) \Longrightarrow$ $\boxed{\mathbf{x + y - 7 = 0}}$

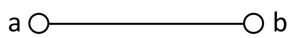
Chapter Three

FUNCTIONS AND THEIR GRAPHS

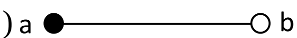
X is a variable symbol, that may take any value in some specified set of numbers. The set of numbers over which (X) may vary is called (domain) of X . We also use the notation X E X to denote that X is an element of set X. The domain of our variables will be intervals of numbers such as the following :

1. The open interval (a , b) (between a & b and not include a & b)

   I = { X : a < X < b }

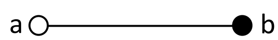
   

2. The half - open interval [a , b) (between a & b and including a)

   I = { X : a ≤ X < b }

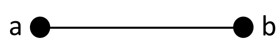
   or : (a , b]

   (between a & b and including b)

   I = { X : a < X ≤ b }

3. The closed interval [a , b] (between a & b and including them)

   I = { X : a ≤ X ≤ b }

Remarks :

- **(- ∞ , 0) all negative real numbers unless zero.**
- **(- ∞ , 0] if zero is also to be included.**
- **(o , ∞) all positive real numbers unless zero.**
- **R or (- ∞ , ∞) all real numbers.**

Example (1): Find the interval of the function $y = \sqrt{1 - x^2}$?

Solution : x^2 must not be greater than 1, because the square root of a negative number is an imaginary :

$1 - x^2 = 0 \Longrightarrow x^2 = 1 \Longrightarrow = \pm 1$, thus the natural domain of x is the closed nterval :

[-1 , 1] , { X : -1 ≤ X ≤ 1 }

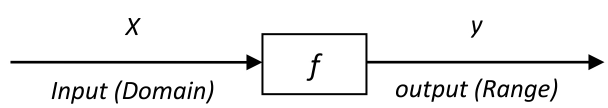
Example (2): Find the interval of the function $y = 1/\sqrt{1-x^2}$?

Solution : the natural domain of x would be the open interval (-1,1) , because we cannot divide by zero (-1 , 1) , { X : -1 < X < 1 }

H.W (1) : Draw the intervals $-1 < X < 5$, $-1 < X \le 5$, $-3 \le X < 2$.

3.1. *Domain and Range* :

The domain and range of many functions are intervals of real numbers. The set of all real numbers that lie strictly between two fixed numbers a and b is an open interval. Thus the input numbers make up the domain of the function and the output make up the function's range.



Example (3): In each of the following functions , the domain is taken to be the largest set of real x – values for which the formula gives real y – values.

Solution :

Function	Domain	Range
$y = x^2$	$-\infty < x < \infty$	$0 \le y$
$y = \sqrt{1-x^2}$	$-1 \le x \le 1$	$0 \le y \le 1$
$y = 1/x$	$x \ne 0$	$y \ne 0$
$y = \sqrt{x}$	$0 \le x$	$0 \le y$
$y = \sqrt{4-x}$	$x \le 4$	$0 \le y$

Note :

x is called independent variable.

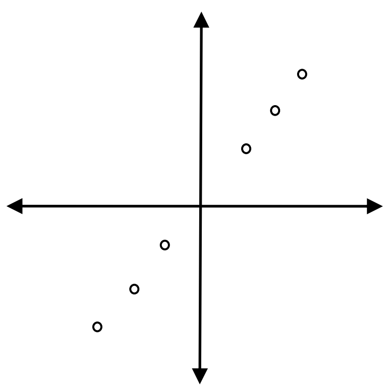
y is called dependent variable.

Example (4): Sketch : 1. y = [x]   2. y = 1 / x   3. y = $|x|$?

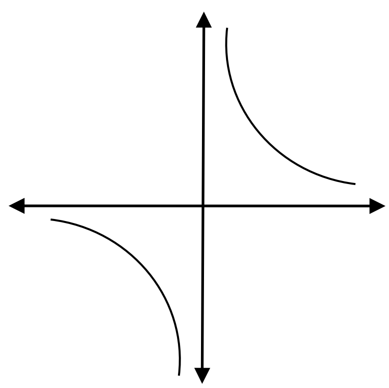
Solution:

1. The square brackets represent the greatest integer. [x] = greatest integer that is x. for example : [-3] = -3 , [-0.3] = -1 , [-2.5] = -3 , [2.5] = 2

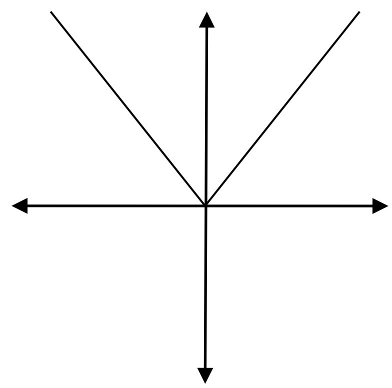
X	1	2	3	-1	-2	-3
Y	1	2	3	-1	-2	-3



(1)



(2)



(3)

Note :

for case (3) y = $|x| = \sqrt{x^2}$ **(absolute value)**

$$\Longrightarrow \begin{cases} x \text{ if } & x \geq 0 \\ -x \text{ if } & x < 0 \end{cases}$$

Example (5): Sketch the function y = $\sqrt{4 - x}$?

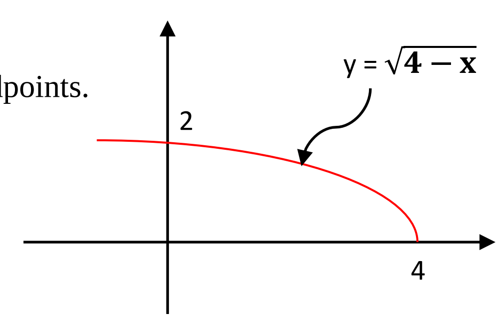
Solution:

Making a table of values that includes intercepts and domain endpoints.

x	4 - x	$\sqrt{4 - x}$
4	0	0
3.75	0.25	0.5
2	2	1.4
0	4	2
-2	6	2.4

Domain : x ≤ 4

Range : 0 ≤ y



3.2. *Even and Odd Functions* :

A function $y = f(x)$ is an Even function of x if :

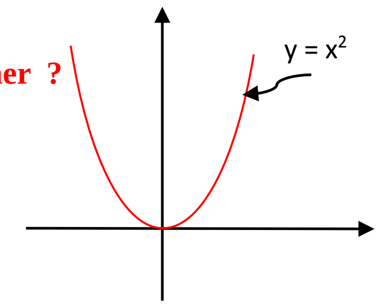
$f(-x) = f(x)$ for every x in the function's domain

it is an Odd function of x if :

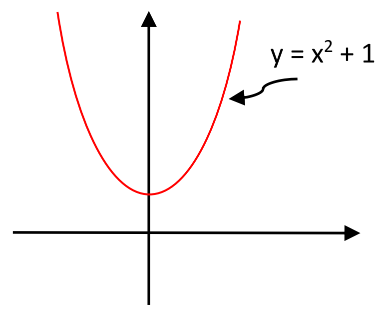
$f(-x) = -f(x)$ for every x in the function's domain

Example (6): Show if the following functions are even , odd or neither ?

Solution:

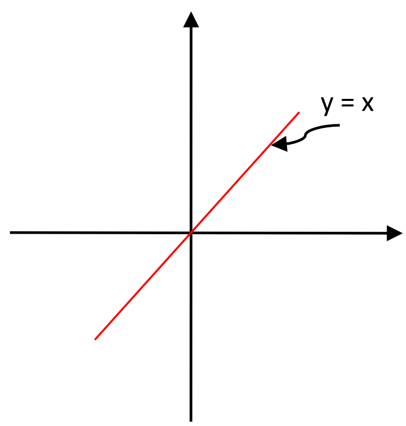


1. $y = x^2$ even function $(-x)^2 = x^2$ and sym. about y – axis.

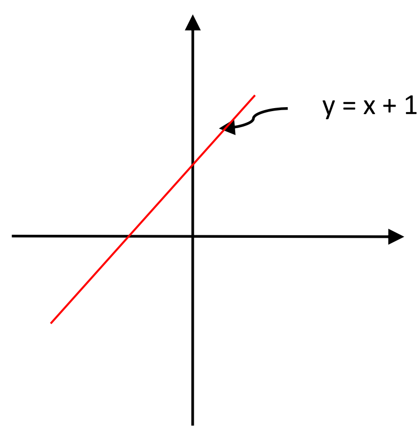


2. $y = x^2 + 1$ even function $(-x)^2 + 1 = x^2 + 1$

   and sym. about y – axis.



3. $y = x$ odd function $(-x) = x$ and sym. about origin.



4. $Y = x + 1$ not odd & not even
   Not sym. about x – axis and not sym. about y – axis

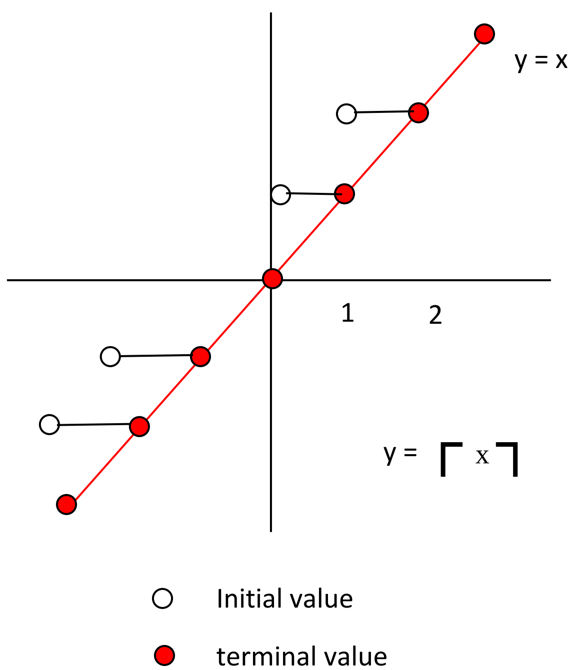
3.3. *Integer - Valued Functions* :

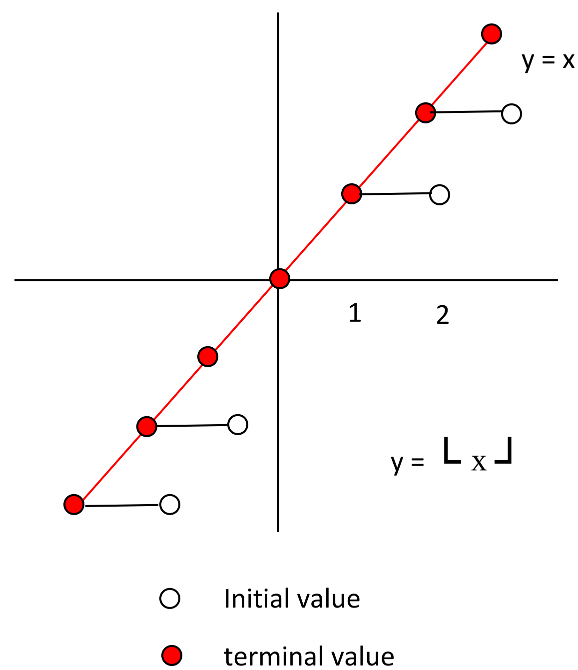
The greatest integer less than or equal to a number x is called (the greatest integer in x). The symbol for it is $\lfloor x \rfloor$, which is read " the greatest integer in x ".

Example (7): **positive :** $\lfloor 1.9 \rfloor = 1$, $\lfloor 2.0 \rfloor = 2$, $\lfloor 2.4 \rfloor = 2$

Zero : $\lfloor 0.5 \rfloor = 0$, $\lfloor 0 \rfloor = 0$

Negative : $\lfloor -1.2 \rfloor = -2$, $\lfloor -0.5 \rfloor = -1$





$y = \lfloor x \rfloor$ is integer floor for x

$y = \lceil x \rceil$ is integer ceiling for x

3.4. Composition of the Functions :

Let f(x) and g(x) are two functions, we can constructed a new functions as follow :

1. Sum [f(x) + g(x)]
2. Product [f(x) . g(x)]
3. Difference [f(x) - g(x)]
4. Quotient [f(x) / g(x)]

The domain of f(x) and g(x) in sum, product and difference is the intersection of domains of f(x) and g(x).

Example (8): find the domain of the function resulted from combining two functions f(x) = $\sqrt{x}$ and g(x) = $\sqrt{1-x}$?

Solution:

Df of f(x) : $x \geq 0$ and Df of g(x) : $x \leq 1$

1. $\sqrt{x} + \sqrt{1-x} = [\,0\,,1\,]$
2. $\sqrt{x} \,.\, \sqrt{1-x} = \sqrt{x\,(1-x)} = [\,0\,,1\,]$
3. $\sqrt{x} - \sqrt{1-x} = [\,0\,,1\,]$
4. $\sqrt{x} \,/\, \sqrt{1-x} = [\,0\,,1\,)$

3.5. Slope of a Curve :

The general equation to find the slope of a curve is :

$m = \Delta y / \Delta x = (y_2 - y_1) / (x_2 - x_1)$

Example (9): find the slope of the curve $y = x^3 - 3x + 3$?

Solution:

$Y_2 = x_2^3 - 3x_2 + 3$ …..(1)

$\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ then $x_2 = \Delta x + x_1$ and $y_2 = \Delta y + y_1$

Substitute the point $(\Delta x + x_1 , \Delta y + y_1)$ in eq. (1) , then :

$\Delta y + y_1 = (\Delta x + x_1)^3 - 3(\Delta x + x_1) + 3$

$\Delta y + y_1 = x_1^3 + 3x_1^2 \Delta x + 3x_1 \Delta x^2 + \Delta x^3 - 3x_1 - 3\Delta x + 3$ ……(2)

Subtracts eq. (1) from (2) : $\Delta y = 3x_1^2 \Delta x + 3x_1 \Delta x^2 + \Delta x^3 - 3\Delta x$

$\Delta y/\Delta x = 3x_1^2 + 3x_1\,\Delta x + \Delta x^2 - 3 = 3x_1^2 - 3 + \Delta x\,(3x_1 + \Delta x)$

Δx approaches to zero ($\Delta x \rightarrow 0$)

$\Delta y/\Delta x = 3x_1^2 - 3 + \Delta x\,(3x_1 + \Delta x)$

$\Delta y/\Delta x = 3x_1^2 - 3$

3.6. *Derivation of a Function* :

$\Delta y = f(x_1 + \Delta x) - f(x_1)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$$

$$\bar{f} = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$$

Therefore; all the expressions $\bar{f}(x)$, dy/dx , dxy, $\bar{y}$ mean the derivative.

Example (10): find $\bar{f}(x)$ of $y = x^3 - 3x + 3$?

Solution:

$$\bar{f} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$f(x + \Delta x) = (x + \Delta x)^3 - 3(x + \Delta x) + 3 = x^3 + 3x^2\,\Delta x + 3x\,\Delta x^2 + \Delta x^3 - 3x - 3\Delta x + 3$

$f(x) = x^3 - 3x + 3$

$f(x + \Delta x) - f(x) = 3x^2\,\Delta x + 3x\,\Delta x^2 + \Delta x^3 - 3\Delta x$

$$\bar{f} = \lim_{\Delta x \to 0} \frac{3x^2\,\Delta x + 3x\,\Delta x^2 + \Delta x^3 - 3\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x\,(3x^2 + 3x\,\Delta x + \Delta x^2 - 3)}{\Delta x}$$

$\therefore \bar{f} = 3x^2 - 3$

3.7. *Velocity and Rate* :

Velocity = distance / time = s / t and the variation of velocity = ds / dt

S = f(t) and $V_{av.} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$

Example (11): find the rate of velocity if S = $\sqrt{t}$?

Solution:

$f(t + \Delta t) = \sqrt{t + \Delta t}$

$f(t) = \sqrt{t}$

$$V_{av.} = \lim_{\Delta t \to 0} \frac{\sqrt{t+\Delta t} - \sqrt{t}}{\Delta t} * \frac{\sqrt{t+\Delta t} + \sqrt{t}}{\sqrt{t+\Delta t} + \sqrt{t}}$$

$$= \frac{t+\Delta t - t}{\Delta t\,(\sqrt{t+\Delta t} + \sqrt{t}\,)} = \boxed{\frac{1}{2\sqrt{t}}}$$

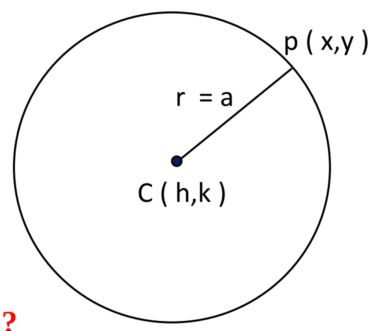
Chapter Four

CONIC SECTIONS

4.1. *Circle* :

If a straight circular cone is taken and cut by a plane perpendicular to its axis we get a circle which is the set of points in a plane whose distance from a given fixed point in the plane is a constant.

$r = a = \sqrt{(x-h)^2 + (y-k)^2}$ or $a^2 = (x-h)^2 + (y-k)^2$



$Ax^2 + By^2 + ax + by + c = 0$

Example (1): find the center and radius of the circle $x^2 + y^2 + 4x - 6y = 12$?

Solution:

$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 12 + 4 + 9$ *(by complete square in x- term & y- term)* [$(4*0.5)^2 = 4$ and $(6*0.5)^2 = 9$]

$(x+2)^2 + (y-3)^2 = 25$ (compare this eq. to the general form of the circle)

$\therefore h = -2 \ , \ k = 3$

$\therefore C(-2,3) \ , \ a = 5$

Example (2): find the center and radius of the circle $2x^2 + 2y^2 + x + y = 0$?

Solution:

$$x^2 + \frac{x}{2} + y^2 + \frac{y}{2} = 0$$

$$\left(x^2 + \frac{x}{2} + \frac{1}{16}\right) + \left(y^2 + \frac{y}{2} + \frac{1}{16}\right) = \frac{1}{8}$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{1}{8}$$

$\therefore C\left(-\frac{1}{4},-\frac{1}{4}\right)$, $a=\frac{1}{2\sqrt{2}}$

Example (3): find an equation for the circle through the points (0,0) and (6,0) that is tangent to the line y = -1 ?

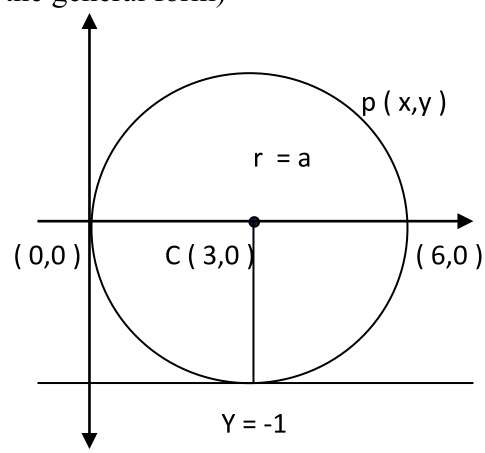
Solution:

$h^2 + k^2 = a^2$ ….. (1) (by substituting point (0,0) in the general form)

$(6-h)^2 + (0-k)^2 = a^2$ ….. (2)

$36 - 12h = 0 \Longrightarrow h = 3$

$(x-3)^2 + (y-4)^2 = 25$



Example (4): find an equation for the circle which is pass through the points (8,0) , (0,6) and (0,0) ?

Solution:

Substitute (8,0) in the eq. :

$64 + 8a + C = 0$ ……..(1)

(0,6) :

$36 + 6b + C + 0$……..(2)

(0,0) :

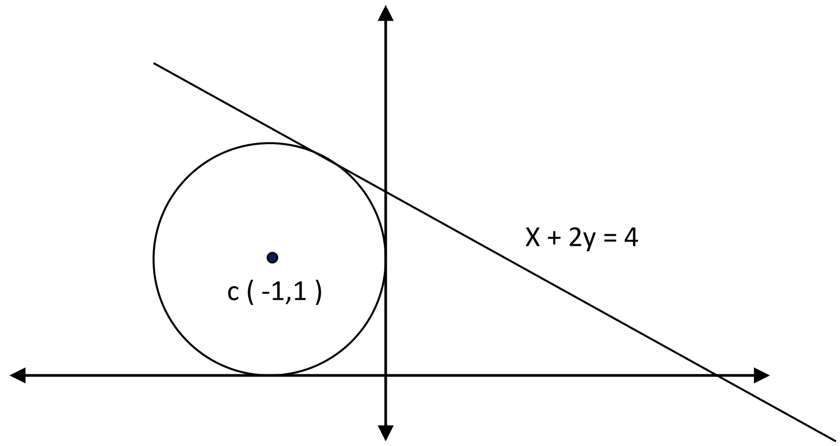
$C = 0$ …. (3) sub. In eq. (1) & (2) :

$a = -8$, $b = -6$

$X^2 + y^2 - 8x - 6y = 0$

Note : number of unknowns equal to number of equations.

Example (5): find an equation for the circle centered at (-1,1) that is tangent to the line x + 2y = 4 ?



Solution:

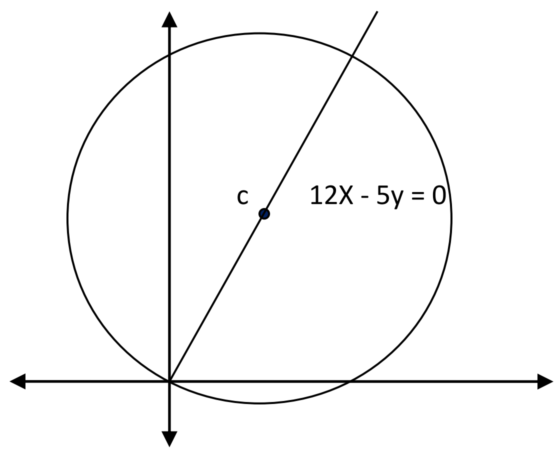
$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

A = 1 , B = 2 , C = -4

$$d = \frac{|1*-1+2*1-4|}{\sqrt{1+4}} = \frac{3}{\sqrt{5}}$$

$$(x+1)^2 + (y-1)^2 = \frac{9}{5}$$

Example (6): find an equation for the circle through the point (0,0) and (17,7) whose center lies on the line 12x – 5y = 0 ?



Solution:

$h^2 + k^2 = a^2$

$(17-h)^2 + (7-k)^2 = a^2$

$289 - 34h + h^2 + 49 - 14k + k^2 = a^2$

$289 - 34h + 49 - 14k = 0$

$-34h - 14k = -338$

$34h + 14k = 338 \ldots (1)$

∵ (h,k) lies on the line , ∴ satisfy the line equation.

$12h - 5k = 0 \Longrightarrow \therefore h = \frac{5k}{12} \quad \ldots. (2)$ sub. In eq. 1

$34\left(\frac{5k}{12}\right) + 14k = 338 \Longrightarrow k = 12 \;\&\; h = 5$ and $a = 13$

$$(x-5)^2 + (y-12)^2 = 169$$

Example (7): if the distance from P(x,y) to the point (6,0) is twice the distance from the point (0,3) , show that P lies on a circle. And find the center and radius.

Solution:

$\sqrt{(x-6)^2+(y-0)^2} = 2\sqrt{(x-0)^2+(y-3)^2}$

$x^2 - 12x + 36 + y^2 = 4\,(x^2 + y^2 - 6y + a)$

$3x^2 + 12x + 3y^2 - 24y = 0$

By completing the square :

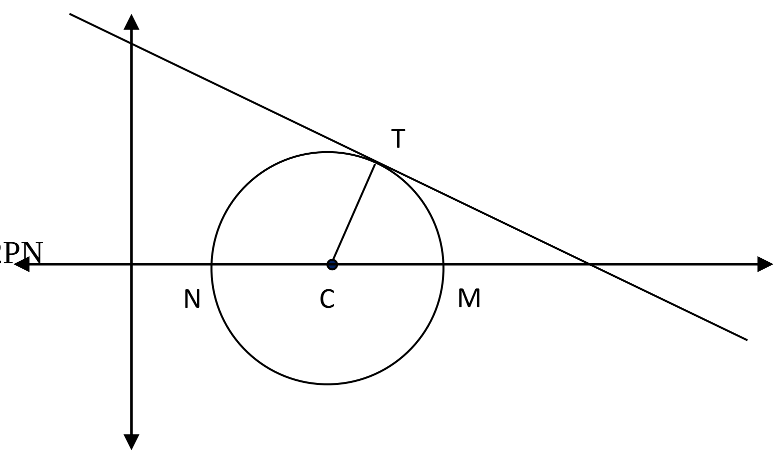
$3(x^2 + 4x + 4) + 3(y^2 - 8y + 16) = 48 + 12$

$3(x + 2)^2 + 3(y - 4)^2 = 60$

$\therefore h = -2 \quad , \quad k = 4 \quad , \quad a = \sqrt{20}$

Example (8) : Let P a point outside a given circle C, let B a tangent to C at T , let the line PN from P through the center of C intersect C at M and N , prove that (PM) (PN) = $(PT)^2$.

Solution:



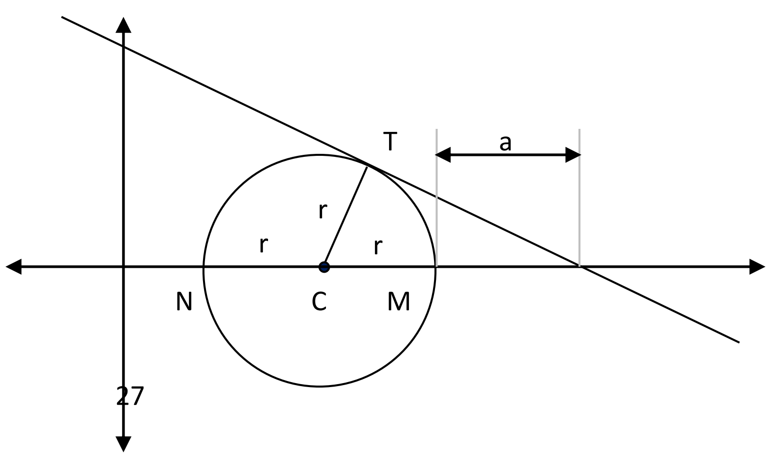
$(PC)^2 = (CT)^2 + (TP)^2$

$\left(\frac{PN+PM}{2}\right)^2 = \left(\frac{PN-PM}{2}\right)^2 + (PT)^2$

$0.25\,(PN^2 + 2PN\,PM + PM^2) = 0.25(PN^2 - 2PN\,PM + PM^2) + 4PT^2 = 4PN\,PM = 4PT^2$

$= PN \,.\, PM = PT^2$

Another solution :



$(PT)^2 + r^2 = (r + a)^2$

$(PT)^2 = (r + a)^2 - r^2 = r^2 + 2ra + a^2 - r^2$

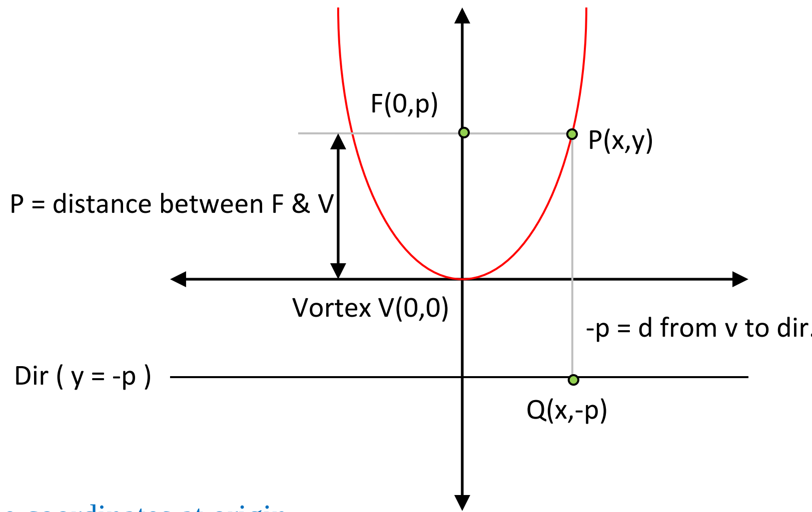
$= a\,(2r + a) = PM.PN$

Note : $a = PM$

$2r + a = PN$

4.2. *Parabola* :

A parabola is the set of points in a plane that are equidistant from a given fixed point and fixed line in the plane. The fixed point is called the focus and the fixed line the directory.



PF = PQ

PF = $\sqrt{x^2 + (y - p)^2}$

PQ = $\sqrt{(y + p)^2}$

$X^2 = 4PY$ Opens Upward and the coordinates at origin.

$X^2 = -4PY$ Opens Downward and the coordinates at origin.

$Y^2 = 4PX$ Opens to right and the coordinates at origin.

$Y^2 = -4PX$ Opens to left and the coordinates at origin.

Example (1) : Find the focus and directory of $x^2 = 8y$.

Solution:

$x^2 = 4py$

$x^2 = 8y \Longrightarrow 4p = 8 \Longrightarrow p = 2$

$\therefore$ F (0 , 2) & Dir. y = -2

4.2.1. *Translating the axes :*

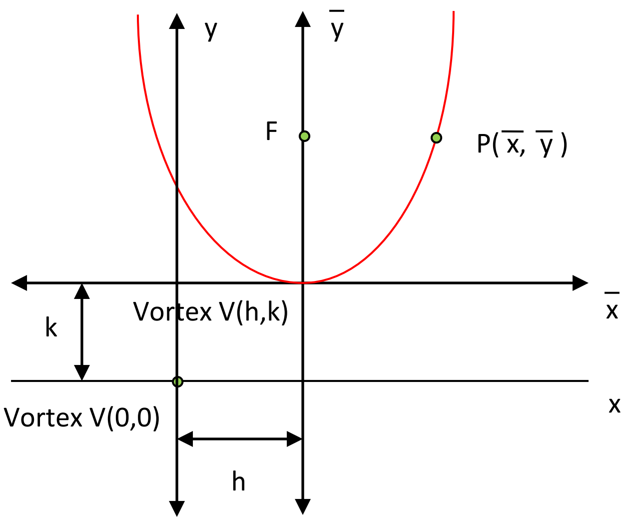
$x = \bar{x} + h$

$y = \bar{y} + k$

$\bar{x} = x - h$

$\bar{y} = y - k$

$\bar{x}^2 = 4p\bar{y}$



$(x - h)^2 = 4p(y - k)$ Upward

$(x - h)^2 = -4p(y - k)$ Downward

$(y - k)^2 = 4p(x - h)$ right

$\bar{y} = -p$, $y = -p + k$

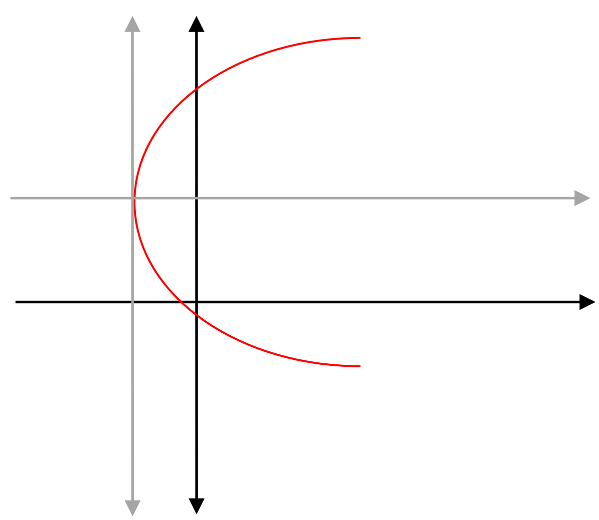
$(y - k)^2 = -4p(x - h)$ left

Example (2) : Find the equation of parabola that have the following data V(-1,2) , F(3,2) .

Solution:

$\because$ V not (0,0) then the axes are translated

From the graph, the parabola opens to the right.



$h = -1$, $k = 2$

$(y - 2)^2 = 4p\,(x + 1)$

Distance between F & V

$$P = \sqrt{(-1-3)^2 + (2-2)^2} = 4$$

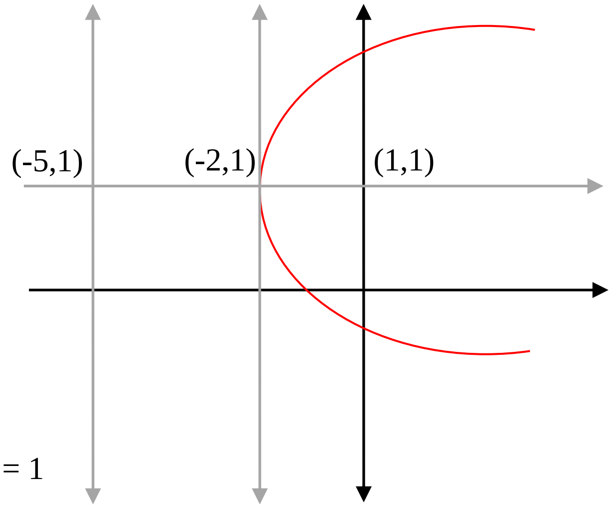
$(y - 2)^2 = 16\,(x + 1)$

Directory is $x = -5$ & axis $y = k \Longrightarrow y = 2$ from:

$\bar{x} = -p$ then $x - h = -p$ & $x = -p + h = -4 - 1 = -5$

Example (3) : $y^2 - 2y - 12x - 23 = 0$, Discuss this equation .

Solution:



$(y - k)^2 = 4p\,(x - h)$

$(y - 1)^2 = 12\,(x + 2)$ where p =3 , k =1 , h = -2

Dir ⟹ (x – h) = -p

x + 2 = -3 ⟹ x = -5

to find d focus :

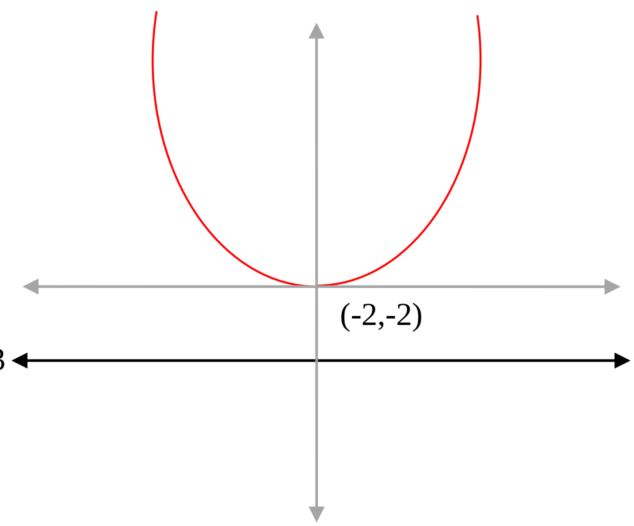
$x_v = (x_1 + x_2) / 2$ ⟹ -2 = (x – 5) / 2 ⟹ x = 1

$y_v = (y_1 + y_2) / 2$ ⟹ 1 = (y + 1) / 2 ⟹ y = 1

focus (1,1)

Example (4) : V(-2,-2) , Dir y = -3 , find eq.

Solution:



h = -2 , k = -2 , $\bar{y} = -p$

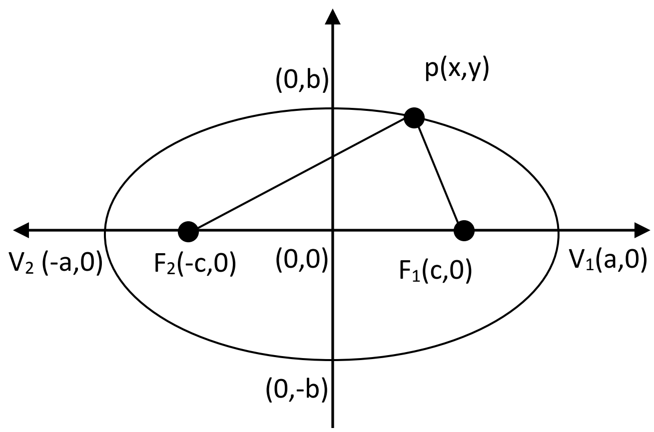
y = -p + k ⟹ p = k – y ⟹ p = - 2 + 3 = 1

$(x - h)^2 = 4p\,(y - k)$

$(x + 2)^2 = 4\,(y + 2)$

4.3. *Ellipses*

An ellipse is the set points in a plane whose distance from two fixed points in the plane have a constant sum.



$PF_1 + PF_2 = 2a \quad , \quad c = \sqrt{a^2 - b^2}$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

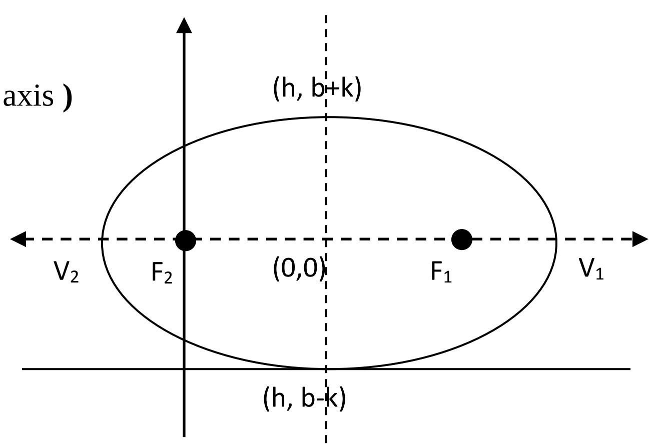
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ (if x – axis is the major axis **)**

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$ (if y – axis is the major axis)

Notes :

1. In ellipse, coefficient of $x^2 \neq y^2$ but they have the same signal.
2. F & V lie on the major axis.

4.3.1. *Center not at the origin :*



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$ (if x – axis is the major axis **)**

$v\,(h \pm a , k)$, $F\,(h \pm \sqrt{a^2 - b^2} , k)$

Referring to figure :

$v_1\,(h+a , k)$ & $v_2\,(h-a , k)$

$F_1(h + \sqrt{a^2 - b^2} , k)$ & $F_2\,(h - \sqrt{a^2 - b^2} , k)$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$ (if y – axis is the major axis **)**

(h, b+k)

$v\,(h , k \pm a)$, $F\,(h , k \pm \sqrt{a^2 - b^2})$

the general equation : $Ax^2 + By^2 + Cx + Dy + E = 0$ where $A \neq B$

Example (1) : find the center , vertices and foci of the ellipse $9x^2 + 4y^2 + 36x - 8y + 4 = 0$?

Solution :

By completing the square :

$9(x^2 + 4x) + 4(y^2 - 2y) = -4$

$9(x^2 + 4x + 4) + 4(y^2 - 2y + 1) = -4 + 36 + 4$

$9(x+2)^2 + 4(y-1)^2 = 36$ (dividing on 36)

$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1$ compare with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

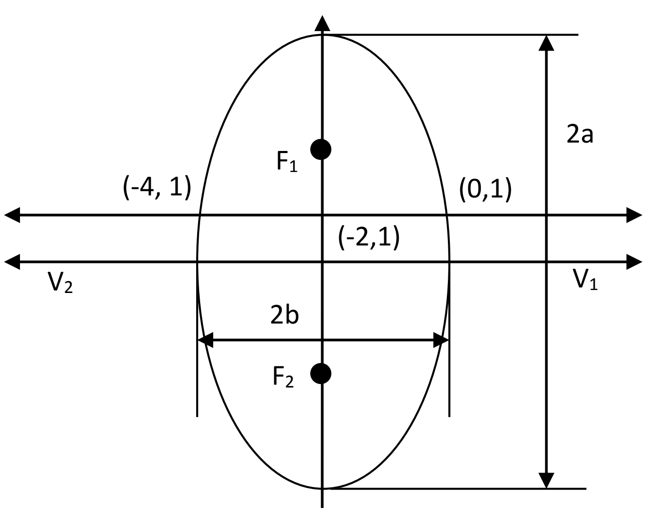
$\because b^2 < a^2 \Longrightarrow a^2 = 9 \Longrightarrow a = 3$

& $b^2 = 4 \Longrightarrow b = 2$

$c = \sqrt{a^2 - b^2} = \sqrt{9-4} = \sqrt{5}$, $h = -2$, $k = 1$ (from comparison)

by applying the rules :

$V_1(-2,4)$, $V_2(-2,-2)$, $F_1(-2, 1+\sqrt{5})$, $F_1(-2, 1-\sqrt{5})$



4.3.2. *Eccentricity (e) :*

c = $\sqrt{a^2 - b^2}$ ⟹ $e = \dfrac{c}{a}$

when

e = 1 conic section is parabola

e < 1 conic section is ellipse

e > 1 conic section is hyperbola

e = 0 conic section is circle

Example (1) : c (0 , 2) , F (0,0) , a = 3 , find equation?

Solution :

h = 0 , k = 2 , a = 3

c = 2 – 0 = 2 (distance between center & focus)

e = 2 / 3 = 0.66 (conic section is ellipse)

b = $\sqrt{a^2 - c^2} = \sqrt{9 - 4} = \sqrt{5}$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \Longrightarrow \frac{x^2}{5} + \frac{(y-2)^2}{9} = 1$$

Homework :

1. C (-3 , 0) , F (-3,-2) , a = 4 , find equation

2. Find the center, vertices and foci of the ellipse $9x^2 + 16y^2 + 18x - 96y + 9 = 0$?

3. Given F_1 (-1,1) , F_2(1,1) , and pass through (0,0) , find eq. ?

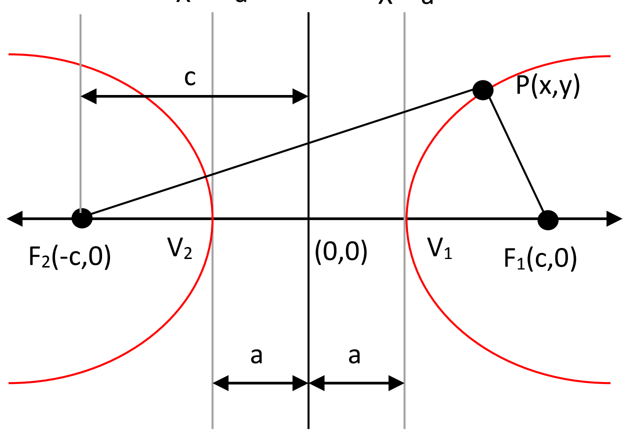
4. Given end points (1,1) , (3,4) , (1,7) , (-1,4) , find equation ?

4.4. *Hyperbolas*

A hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$ (equation of hyperbola having vertices, foci & center on the x – axis)

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$ (equation of hyperbola having vertices, foci & center on the y – axis)



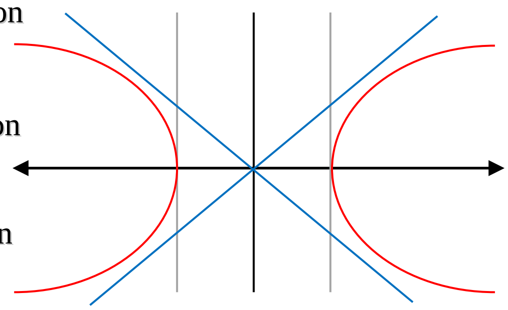
4.4.1. *Asymptotes of Hyperbolas*

$y=\frac{b}{a}\,x$ asymptotes with respect to x – axis for positive direction

$y=-\,\frac{b}{a}\,x$ asymptotes with respect to x – axis for negative direction

$y=\frac{a}{b}\,x$ asymptotes with respect to y – axis for positive direction

$y=-\,\frac{a}{b}\,x$ asymptotes with respect to y – axis for negative direction



4.4.2. *Center not at the origin*

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$ when lines of foci parallel to x – axis

Vertices : (h ± a , k)

Foci : ($h \pm \sqrt{a^2+b^2}$, k)

Asymptotes : (y - k) = $\pm \frac{b}{a}\,(x - h)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$ when lines of foci parallel to y – axis

Vertices : (h , k ± a)

Foci : (h , k ± $\sqrt{a^2 + b^2}$)

Asymptotes : (y - k) = $\pm \frac{a}{b}\,(x - h)$

Note :

To recognize the type of conic section :

- If the coefficient of $x^2 = y^2$ (magnitude & signal) , then the conic section is circle.
- If the coefficient of x^2 have the same signal of the coefficient of y^2 and not equal in the magnitude , then the conic section is ellipse.
- If the coefficient of x^2 equal or not equal to the coefficient of y^2 and they have different
- Unless above , the conic section is parabola.

Example (1) : Find the center, vertices, foci and asymptotes of the hyperbola $4x^2 - y^2 + 8x + 2y - 1 = 0$?

Solution :

$4x^2 - y^2 + 8x + 2y - 1 = 0$, $4(x^2 + 2x) - (y^2 - 2y) = 1$
$4(x^2 + 2x + 1) - (y^2 - 2y + 1) = 1$
$4(x + 1)^2 - (y - 1)^2 = 4$
$$\frac{(x+1)^2}{1} - \frac{(y-1)^2}{4} = 1$$
∴ h = -1 , k = 1 , c (-1 , 1) , a = 1 , b = 2

$V_1(h \pm a,\ k) \Longrightarrow V_1(-2,1)$ & $V_2(0,1)$

$F_1(h \pm \sqrt{a^2 + b^2},\ k) \Longrightarrow F_1(-1-\sqrt{5},\ 1)$, $F_1(-1+\sqrt{5},\ 1)$

$(y-1) = \pm 2(x + 1) \Longrightarrow y = 2x + 3$ & $y = -2x - 1$ (solving simultaneously to find the cross point) :

y = 1 & x = - 1

for check : $e = c / a = \sqrt{5} / 1 = \sqrt{5} > 1$ (hyperbola)

Chapter Five

LIMITS

Suppose we take the function $y = x^2 + 1$ we want to know the limit of this function. If the point P (x,0) move on the axis and the point Q(x, f(x)) move on the curve.
If P tends to origin point and approach to zero, then Q sliding to point (0 , 1) ; therefore ; f (x) approach to (1) , we write :

$$\lim_{x\to 0} f(x) = 1$$

$$\lim_{x\to +0} f(x) = 1 \quad \text{(from the right)}$$

$$\lim_{x\to -0} f(x) = 1 \quad \text{(from the left)}$$

Example (1) : Show that f (x) = $|a|$ has no derivatives at x = 0?

Solution :

$$\lim_{x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{n\to\infty}\frac{|a+\Delta x|-|a|}{\Delta x} = \lim_{n\to\infty}\frac{|a|+|\Delta x|-|a|}{\Delta x} = \lim_{n\to\infty}\frac{|\Delta x|}{\Delta x}$$

$$\lim_{x\to +0}\frac{\Delta x}{\Delta x} = 1 \quad \text{(from the right)}$$

$$\lim_{x\to -0}\frac{-\Delta x}{\Delta x} = -1 \quad \text{(from the left)}$$

The right and left hand are not equal; therefore; there is no limit and no derivative.

Note :

The function which have a derivative have a limit and the opposite is not correct.

5.1. The Limit Combination Theorem :

Let $\lim_{x \to a} f(x) = L_1$; $\lim_{x \to a} g(x) = L_2$; L_1 & L_2 real numbers

1. $\lim_{x \to a}[f(x) \mp g(x)] = \lim_{x \to a} f(x) \mp \lim_{x \to a} g(x) = L_1 \mp L_2$

2. $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = kL_1$ **(k is constant)**

3. $\lim_{x \to a}[f(x).g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x) = L_1 = L_1.L_2$

4. $\lim \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$ **(where $L_2 \neq 0$)**

5. $\lim_{x \to a} C = C$

Example (2) : **Find limit** $\lim_{x \to 2}(5x^5 - 13x^2 + 10)$

Solution :

$\lim_{x \to 2}(5x^5 - 13x^2 + 10) = \lim_{x \to 2}(5x^5) + \lim_{x \to 2}(-13x^2) + \lim_{x \to 2}(10)$

$= 5 \lim_{x \to 2}(x^5) - 13 \lim_{x \to 2}(x^2) + \lim_{x \to 2}(10)$

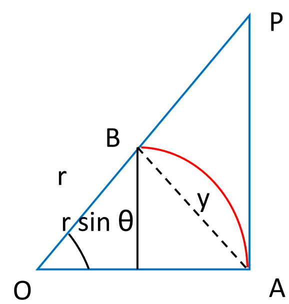
$= 5 * 2^2 – 13 * 2^2 + 10 = 118$

5.2. The Limits of Trigonometric Functions :

1. $\lim_{\theta \to 0} \sin\theta = 0$

2. $\lim_{\theta \to 0} \cos\theta = 1$

Example (3) : Prove that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$



Solution :

AB < sector AB < AP

Δ AOB < sector AOB < Δ AOP

Δ AOB = 0.5 * r * r sin θ = 0.5 r^2 sin θ

Sector AOB = (θ/2π) * π r^2 = 0.5 r^2 θ

Δ AOP = 0.5 * r * r tan θ = 0.5 r^2 tan θ

0.5 r^2 sin θ < 0.5 r^2 θ < 0.5 r^2 tan θ

Sin θ < θ < tan θ

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

When θ → 0 then cos θ = 1

$1 < \frac{\theta}{\sin\theta} < 1$

Now $\frac{\theta}{\sin\theta}$ lies between 1 and quantity which tends to 1.

$\therefore \frac{\theta}{\sin\theta}$ also tends to 1

$\therefore \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Example (4) : Find limit $\lim_{x \to 0} \frac{\sin 3x}{x}$

Solution :

$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{3x \to 0} \frac{3\sin 3x}{3x}$ (by multiplying numerator & denominator * 3)

$= 3 * 1 = 3$ (because $\frac{\sin 3x}{3x} = 1$)

Example (5) : Find limit $\lim_{x \to 0} \frac{\tan x}{x}$

Solution :

$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} [\frac{\sin x}{x} * \frac{1}{\cos x}] = \lim_{x \to 0} \frac{\sin x}{x} . \lim_{x \to 0} \frac{1}{\cos x} = 1 * 1 = 1$

Example (6) : Find limit $\lim_{x \to 2} \frac{x^2 - 4}{x-2}$

Solution :

$\lim_{x \to 2} \frac{x^2 - 4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = 2 + 2 = 4$

Example (7) : Find limit $\lim_{\theta \to \pi/2} \tan\theta - \sec\theta$

Solution :

$\lim_{\theta \to \pi/2} \frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta} = \lim_{\theta \to \pi/2} \frac{\sin\theta - 1}{\cos\theta} = \lim_{\theta \to \pi/2} \frac{\cos\theta}{-\sin\theta} = 0$

Example (8) : Find limit $\lim_{\propto \to \pi/2} \frac{\cos \propto}{\frac{\pi}{2} - \propto}$

Solution :

Let $\theta = (\pi / 2) - \alpha$

$\cos \alpha \to (\pi / 2)$ & $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\cos(\frac{\pi}{2} - \theta)}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Example (9) : **Find limit** $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$

Solution: $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} * \frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} * \frac{\sin\theta}{\theta(1+\cos\theta)} = 1 * \frac{0}{1+1} = 0$

Example (10) : **Find limit** $\lim_{x \to 0} (\sin x)^x$

Solution :

Let $y = (\sin x)^x$

$\ln y = x \ln \sin x$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln \sin x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x \to 0} (-x^2 \cot x) = \lim_{x \to 0} \frac{-x^2}{\tan x}$$

$$= \lim_{x \to 0} \frac{-2x}{\sec^2 x} = 0$$

Example (11) : **Find limit** $\lim_{\theta \to 0} \frac{\sec 2\theta \tan 3\theta}{5\theta}$

Solution :

$$\lim_{\theta \to 0} \frac{\frac{1}{\cos 2\theta} * \frac{\sin 3\theta}{\cos 3\theta}}{5\theta} = \lim_{\theta \to 0} \frac{\sin 3\theta}{5\theta(\cos 2\theta . \cos 3\theta)} = \lim_{\theta \to 0} \frac{\frac{\sin 3\theta}{3\theta}}{\frac{5\theta}{3\theta}(\cos 2\theta . \cos 3\theta)} = \frac{3}{5}$$

Example (11) : **Find limit** $\lim_{y \to \frac{\pi}{2}} \frac{\cos y}{\frac{\pi}{2} - y}$

Solution :

Let $A = (\pi / 2) - y \Longrightarrow y = (\pi / 2) - A$

$$\lim_{y \to \frac{\pi}{2}} \frac{\cos y}{\frac{\pi}{2} - y} = \lim_{A \to 0} \frac{\cos(\frac{\pi}{2} - A)}{A} = \lim_{A \to 0} \frac{\sin A}{A} = 1$$

Example (12) : Find limit $\lim_{x\to 0} \frac{\tan 5x}{\sin 4x}$

Solution : $\lim_{x\to 0} \frac{\frac{\sin 5x}{\cos 5x}}{\sin 4x} = \lim_{x\to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 4x.\cos 5x}{5x}} = \lim_{x\to 0} \frac{\frac{5x}{4x}}{\frac{\sin 4x.\cos 5x}{4x}} = \lim_{x\to 0} \frac{x}{\cos 5x} = \frac{0}{1} = 0$

Home work :

1. **Find limit** $\lim_{x\to 2} \frac{x^2-5x+6}{x-2}$
2. **Find limit** $\lim_{x\to a} \frac{x^3-a^3}{x^4-a^4}$
3. **Find limit** $\lim_{x\to 0} \frac{1+\sin x}{\cos x}$
4. **Find limit** $\lim_{t\to 0} \frac{2\sin t \cos t}{t}$
5. **Find limit** $\lim_{x\to \pi/2} (\sec x - \tan x)$
6. **Find limit** $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$
7. **Find limit** $\lim_{\alpha\to \pi/2} \frac{\cos\alpha}{\frac{\pi}{2}-\pi}$

5.4. Infinity as a Limit :

To calculate the limit when x → ∞ there are two ways :

1. Divide the numerator and denominator by largest power of x in the denominator and watch what happens to the new numerator and denominator as x → ∞.

2. Another way is to let x = 1/n and calculate the limit as n → +0 if x → + ∞ and as n → --0 if x → -∞.

Note :

1. $\lim_{x\to\infty} \frac{1}{x} = 0$
2. $\lim_{x\to 0} \frac{1}{x} = \infty$

Example (13) : Find limit $\lim_{\theta\to 0} \frac{1-\sec^2\theta}{\sin^2\theta}$

Solution :

$$\lim_{\theta\to 0}\frac{1-sec^2\theta}{sin^2\theta}=\lim_{\theta\to 0}\frac{-tan^2\theta}{sin^2\theta}=-\lim_{\theta\to 0}\frac{sin^2\theta}{cos^2\theta}\cdot\frac{1}{sin^2\theta}=-1$$

Example (14) : **Find limit** $\lim_{x\to\infty}\frac{5x+3}{2x^2-1}$

Solution : Let x = 1 / n , n → 0 and x → ∞

$$\lim_{n\to 0}\frac{\frac{5}{n}+3}{\frac{2}{n^2}-1}=\lim_{n\to 0}\frac{\frac{5+3n}{n}}{\frac{2-n^2}{n^2}}=\lim_{n\to 0}\frac{5n+3n^2}{2-n^2}=\frac{0}{2}=0$$

Or by the first method :

$$\lim_{x\to\infty}\frac{\frac{5}{x}+\frac{3}{x^2}}{2-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{5x+3}{x^2}*\frac{x^2}{2x^2-1}=\frac{0+0}{2-0}=0$$

Example (15) : **Find limit** $\lim_{x\to\infty}x\sin\frac{1}{x}$

Solution: ∵ the same power, we cannot divide, then we can solve this problem by the second way.

Let x = 1 / n , n → 0 and x → ∞

$$\lim_{n\to 0}\sin\frac{n}{n}=1$$

Example (16) : **Find limit** $\lim_{x\to\infty}\frac{\sqrt{x+1}}{\sqrt{4x-1}}$

Solution:

$$\lim_{x\to\infty}\frac{\sqrt{x+1}}{\sqrt{4x-1}}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x}}}{\sqrt{4-\frac{1}{x}}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$$

Home work :

1. **Find limit** $\lim_{x\to -\infty}\frac{2x^2-3}{7x+4}$

2. **Find limit** $\lim_{x \to -\infty} \frac{\sin x}{|x|} = \frac{\sin(-x)}{|-x|} = \frac{\sin -x}{x} = \frac{-\sin x}{x} = -1$
3. **Find limit** $\lim_{x \to \infty} (2 + \frac{\sin x}{x})$
4. **Find limit** $\lim_{x \to \infty} \sqrt{x}$

Chapter Six

CONTINUITY

The function y = f(x) is continuous at x = c if and only if all three of the following statements are true :

1. f (c) exists (c is in the domain of f)
2. $\lim_{x \to 0} f(x)$ exists (f has a limit as x → c)
3. $\lim_{x \to 0} f(x)$ = f (c) (the limit equals the function value)

Example (1) : is the following function is continuous at f (2) , f (x) = $\frac{x^2+x-6}{x^2-4}$

Solution:

f (2) = $\frac{4+2-6}{4-4} = \frac{0}{0}$ there is no limit and not continuous

we will make it continuous :

f (x) = $\frac{x^2+x-6}{x^2-4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x+3}{x+2} = \frac{2+3}{2+2} = \frac{5}{4}$

$$f(x) = \begin{cases} \frac{x^2+x-6}{x^2-4} & x \neq 2 \\ \frac{5}{4} & x = 2 \end{cases}$$

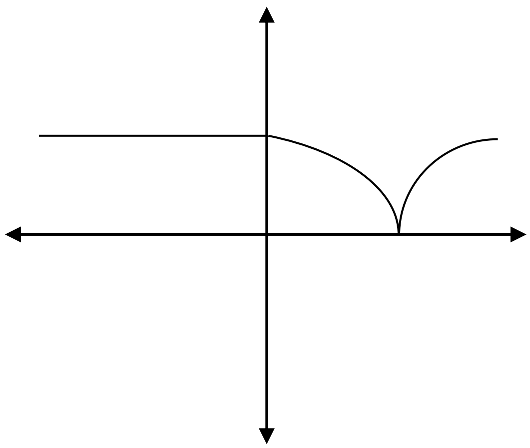
It is continuous at x = 2 because $\lim_{x\to 2}$ f(x) exists = f (x)

The above function is called continuous extension to the original function to the point x = 2 .

Example (2) : is the following function is continuous at x = 0 and x = 1

Solution:

$$f(x) = \begin{cases} 1 & \text{for } x < 0 \\ \sqrt{1-x^2} & \text{for } 0 \le x \le 1 \\ x - 1 & \text{for } x > 1 \end{cases}$$



a. at x = 0 , f (0) = 1

$\lim_{x\to -0} f(x) = \lim_{x\to -0} 1 = 1$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sqrt{1-x^2} = 1$ ∴ $\lim_{x\to 0} f(x) = f(0)$ is continuous

b. at x = 1

f(1) = 0

$\lim_{x\to -1} f(x) = \lim_{x\to -1} \sqrt{1-x^2} = 0$

$\lim_{x\to +1} f(x) = \lim_{x\to +1} x - 1 = 0$

∴ $\lim_{x\to 1} f(x) = f(1)$ is continuous

Example (3) : is the following function is continuous @ x = 0

Solution:

y = sin x / x is not continuous @ x = 0 , but $\lim_{x\to 0} \sin\frac{x}{x} = 1$

$$f(x) = \begin{cases} \sin x / x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Which is continuous at x = 0 because $f(0) = \lim_{x\to 0} f(x)$

Example (4) : is the following function is continuous @ x = -1 & x = 1

Solution:

$$f(x) = \begin{cases} -3x - 4 & \text{for } x < -1 \\ x & \text{for } -1 \le x \le 1 \\ (x-2)^2 & \text{for } x > 1 \end{cases}$$

a. x = - 1

f (-1) = -1

L.L. = $\lim_{x\to -1} f(x) = \lim_{x\to -1} f(-3x-4) = -1$

R.L. = $\lim_{x\to 0} f(x) = \lim_{x\to 0} (x) = -1$

∴ L.L. = R.L. = f (-1)

∴ f (x) is continuous at x = -1

b. x = 1

f(1) = 1

L.L. = $\lim_{x\to -1} f(x) = \lim_{x\to -1} (x) = 1$

R.L. = $\lim_{x\to 0} f(x) = \lim_{x\to 0} (x-2)^2 = 1$

∴ L.L. = R.L. = f (1)

∴ f (x) is continuous at x = 1

Example (5) : is the following function is continuous @ x = -2 & x = 5

Solution:

$y = \frac{x+3}{x^2 - 3x+10}$

$y = \frac{x+3}{(x+2)(x-5)}$

∴ y is discontinuous at x = -2 & x = 5 (because this values make the denominator equal to zero).

Example (6) : find (a) & (b) so that f(x) is continuous at any point of domain and draw ?

Solution:

$$f(x) = \begin{cases} 3x^2 + 2x - 1 & \text{for } 0 \le x < 3 \\ 5x - 3a & \text{for } 3 \le x \le 5 \\ 2x^2 - b & \text{for } x > 5 \end{cases}$$

@ x = 3

f(3) = 5 (3) – 3 (a) = 15 – 3a .... (1)

$\lim_{x \to -3} f(x) = \lim_{x \to -3} (3x^2 + 2x - 1) = 32$

$\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - b) = 2(3)^2 - b = 18 - b$ .... (2)

If f(x) becomes continuous at any point, the following condition must be satisfied.

F(3) = L.L. = R.L.

15 – 3a = 32 .... (1)

From (1) 3a = 15 – 32 ⟹ a = -17/3

From (2) b = 18 – 32 ⟹ b = - 14

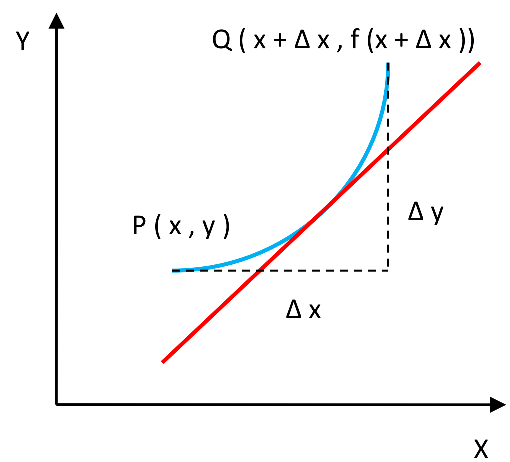
Chapter Seven

DERIVATIVES

m secant = $\frac{\Delta y}{\Delta x}$ = slope of tangent

y + Δ y = f (x + Δ x) $\Longrightarrow$ Δ y = f (x + Δ x) – f (x)

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \bar{f}(x)$$



Rule (1) : *The derivative of a constant is zero.*

$$\frac{\Delta \mathbf{y}}{\Delta \mathbf{x}} (\mathbf{c}) = \mathbf{0}$$

Where c is a constant , for example :

$\frac{\Delta y}{\Delta x}\, 5 = 0$

Note : $\bar{y}$, dy/dx , $\bar{f}\,(x)$ all these expressions mean derivative.

Rule (2) : *Power rule for positive integer powers of x if n is positive integer , then :*

$$\frac{d}{dx} x^n = n\, x^{n-1}$$

For example :

y = x^4 , then $\bar{y} = 4x^3$

Rule (3) : *The(constant) multiple rule : if U is any differentiable function of x and c is any constant , then :*

$$\frac{d}{dx}(c\,u) = c\,\frac{du}{dx}$$

$$\frac{d}{dx}(c\,x^n) = c\,n\,x^{n-1}$$

For example :

y = 5 x^4 , then $\bar{y} = 5 * 4\,x^3 = 20\,x^3$

Example (1) : The line y = 3x + b is tangent to the curve y = $2x^2$, find b and the point of tangency ?

Solution:

$\because y = mx + b$ or :

Y – 3x – b = 0 , $m = \frac{-A}{B} = \frac{3}{1} = 3$

m tangent = 3

dy / dx = 4x $\Longrightarrow$ 3 = 4x $\Longrightarrow$ x = 3 / 4

y = $2x^2$ $\Longrightarrow$ y = 2 $(3/4)^2$ = 9 / 8

P $(\frac{3}{4}, \frac{9}{8})$ $\Longrightarrow$ $\frac{9}{8} = 3 * \frac{3}{4} + b$ $\Longrightarrow$ $b = -\frac{9}{8}$

$\therefore y = 3x - \frac{9}{8}$

Rule (4) : *The sum rule u ,v is differentiable function :*

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Example (2) : Find the derivative of y = $x^4 + 3x^3 - 2x^2 + 4x + 20$?

Solution: $\bar{y} = 4x^3 + 9x^2 - 4x + 4$ & $\bar{\bar{y}} = 12x^2 + 18x - 4$ & $y^{(3)} = 24x + 18$ & $y^{(4)} = 24$ & $y^{(5)} = 0$

Rule (5) : *The product rule (the product of two differentiable functions u & v :*

$$\frac{d}{dx}(u\,v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Rule (6) : *Positive integer power of differentiable function :*

$$\frac{d}{dx}(u)^n = n\,u^{n-1}\frac{du}{dx}$$

Example (3) : Find the derivative of y = $(x^2 - 3x + 1)^5$?

Solution:

$dy / dx = 5(x^2 - 3x + 1)^4 * (2x - 3)$

Rule (7) : *The quotient rule :*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Where $v \neq 0$

Example (4) : Find the derivative of y = $\frac{x^2+1}{x^2-1}$?

Solution:

dy / dx = $\frac{[(x^2-1)*2x]-[(x^2+1)*2x]}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$

Rule (8) : *Negative integer power of differentiable function y = x^n (n is negative) :*

$$\frac{d}{dx}(u^n) = n\,u^{n-1}\frac{du}{dx}$$

Example (5) : Find the derivative of $y = (\frac{2x-1}{x+7})^3$ **?**

Solution:

dy / dx = $3(\frac{2x-1}{x+7})^2 * \left[\frac{[(x+7)*2]-[1*(2x-1)]}{(x+7)^2}\right] = 45 * \frac{(2x+1)^2}{(x+7)^2}$

Example (6) : Do not use the quotient rule to find the derivative of $y = \frac{(x-1)(x^2-2x)}{x^4}$ **?**

Solution:

$y = \frac{(x-1)(x^2-2x)}{x^4}$

$y = (x-1)(x^2-2x) * x^{-4} = (x^{-3} - x^{-4})(x^2-2x)$

dy / dx = $(x^{-3} - x^{-4}) * 2x - 2 + (x^2-2x) * -3x^{-4} + 4x^{-5}$

dy / dx = $-\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$

Example (7) : A heavy rock blasted vertically upward with a velocity of 160 ft / sec. reaches a height of s = 160 t – 16 t^2:

a. **How a height does the rock go?**
b. **V= ? at s = 256 ft.**

Solution:

a. $\bar{s} = v = 160 - 32t = 0 \Longrightarrow$ **t = 160 / 32 = 5 sec.**
s = 160 * 5 – 16 * (5) ^{2} = 400

b. 16 t^2 - 160 t – 256 = 0 $\Longrightarrow$ 16 (t^2 – 10 t – 16) = 0 $\Longrightarrow$ 16 (t – 8) (t – 2) = 0
t = 8 & t = 2 :
v = 160 – 32 t = 160 – 32*2 = 96 m/s.
v = 160 – 32 * 8 = - 96 m/s. (neglected)

7.1. Implicit Differentiation and Fractional Powers :

Some equations cannot be solved to give y explicitly in terms of x , i.e. f (x , y) = 0

Example (8) : Find $\bar{y}$ if $x^5 + 4\,x\,y^3 + y^5 = 2$?

Solution:

$$\frac{d}{dx}\,(x^5) + \frac{d}{dx}\,(4xy^3) + \frac{d}{dx}\,(y^5) = 2$$

$$5x^4 + 4(3\,x\,y^2 + y^3 * 1) + 5y^4\,\frac{dy}{dx} = 0$$

$$(12\,x\,y^2 + 5\,y^4)\,\frac{dy}{dx} + 5\,x^4 + 4\,y^3 = 0$$

$$\frac{dy}{dx} = \frac{-(5\,x^4 + 4y^3)}{12x\,y^2 + 5\,y^4}$$

Example (9) : Find $\bar{\bar{y}}$ if $2x^3 - 3y^2 = 7$?

Solution:

$$6x^2 - 6\,y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x^2}{y} \Longrightarrow \frac{d^2y}{dx^2} = \frac{2yx - x^2 * \frac{dy}{dx}}{y^2} = \frac{2yx - x^2 * \frac{x^2}{y}}{y^2} = \frac{2x}{y} - \frac{x^4}{y^3}$$

7.2. Fractional Powers :

Rule (8) : *Power Rule for fractional exponents:*

$$\frac{d}{dx}\left(u^{p/q}\right) = \left(\frac{p}{q}\right) u^{\left(\frac{p}{q}\right)-1} \frac{du}{dx}$$

Example (10) : **Find $\bar{y}$ & $\bar{\bar{y}}$ if $x^{2/3} + y^{2/3} = 1$?**

Solution:

$$\frac{2}{3} x^{-\frac{1}{3}} - \frac{2}{3} y^{-\frac{1}{3}} * \frac{dy}{dx} = 0 \Longrightarrow \frac{2}{3x^{1/3}} + \frac{2}{3\,y^{1/3}} * \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3} x^{-\frac{2}{3}} * y^{-\frac{1}{3}} + \frac{1}{3} x^{-\frac{4}{3}} * y^{\frac{1}{3}}$$

Example (11) : **Find : 1. the slope of the curve $x^2 + xy + y^2 = 7$ at the point (1, 2)**

2. the tangent point parallel to x – axis ?

3. the tangent point parallel to y – axis ?

Solution:

1. $x^2 + xy + y^2 = 7 \Longrightarrow 2x + x * \frac{dy}{dx} + y + 2y * \frac{dy}{dx} = 0$

$(x + 2y) * \frac{dy}{dx} + (2x + y) = 0$

$\therefore \frac{dy}{dx} = \frac{-(2x+y)}{x+2y} = -4/5$ (at point (1,2))

2. slope dy / dx = 0 (because it's parallel to x – axis)

$2x + y = 0 \Longrightarrow y = -2x \Longrightarrow x^2 + x(-2x) + 4x^2 = 7$

$x = \pm\sqrt{7/3} = \pm 1.53$, $y = \pm 0.76$

3. slope $dy / dx = \infty$ (because it's parallel to y – axis)

$x + 2y = 0$ $x = -2y$ $4y^2 - 2y^2 + y^2 = 7$

$y = \pm\sqrt{7/3} = \pm 1.53$, $x = \pm 3.06$

Home Work : Find the tangent of the curve $\sqrt{\frac{x}{y^3}} - 2\sqrt{\frac{x^3}{y}} = 0$ in the point ($\frac{1}{2}$, 1) ?

7.3. Normal Lines :

The normal line is the line perpendicular to the tangent.

Example (12) : Find the lines tangent and normal to the $y^2 - 6x^2 + 4y + 19 = 0$ at (2,1) ?

Solution:

$2y\frac{dy}{dx} - 12x + 4\frac{dy}{dx} = 0$

$(2y + 4)\frac{dy}{dx} = 12x$

$\frac{dy}{dx} = \frac{12x}{2(y+2)} = \frac{6x}{y+2} = 4$ (by substituting (2,1))

$y - 1 = 4(x - 2) \Longrightarrow y = 4x - 7$

and the slope of normal line = - 1/ 4

$y - 1 = (-1/4)(x - 2) \Longrightarrow y = -0.25x + 1.5$

7.4. The Chain Rule :

The rule for calculating the derivative of the composite of two differentiable functions is that the derivative of their composite is the product of their derivatives.

$$y = g(x) , \quad x = f(t)$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

y = g o f (that mean using the chain rule)

$\bar{\mathbf{y}} = \bar{\mathbf{g}}\,[\,\mathbf{f}(\mathbf{t})\,]\,.\,\bar{\mathbf{f}}\,(\mathbf{t})$

Example (13) : **Find dy/dt if $y = x^3 - 3$ and $x = f(t) = t^2 - 1$, find $\frac{d}{dt}$ (gof) when t = 2 ?**

Solution:

$dy/dx = 3x^2$, $dx/dt = 2t$ ⟹ $dy/dt = 3x^2 \cdot 2t = 6t\,(t^2 - 1)^2 = 108$

Example (14) : **Find dy/dt if $g(x) = \sqrt{x+2}$ and $x = f(t) = t^3 - 1$, find $\frac{d}{dt}$ (gof) when t = 2 ?**

Solution:

$f(2) = 2^3 - 1 = 7$

gof at t = 2 = $\bar{g}$ at x = f(2) . $\bar{f}$ at t=2

$\bar{g}$ at x = f(2) = $\frac{1}{2(x+2)^{\frac{1}{2}}} = \frac{1}{2(7+2)^{\frac{1}{2}}} = \frac{1}{6}$

$\bar{f}$ at t = 2 = $3t^2 = 3*2^2 = 12$

$\therefore\ \overline{(gof)}$ at t = 2 = $\frac{1}{6} * 12 = 2$

Another solution :

$$\frac{dy}{dx} = \frac{1}{2(x+2)^{\frac{1}{2}}} = \frac{1}{2(t^3-1+2)^{\frac{1}{2}}} = \frac{1}{6}$$

$$\frac{dx}{dt} = 3t^2 = 3*4 = 12$$

$$\therefore\ \frac{dy}{dt} = \frac{1}{6} * 12 = 2$$

Example (15) : **Find d/dt (gof) at given value of t = -1 if $g(x) = (\frac{x-1}{x+1})^2$ and $x = f(t) = \frac{1}{t^2} - 1$?**

Solution: x = 1/1 – 1 = 0

$\overline{(gof)}$ at t = -1 = $\bar{g}$ at x = f(-1) * $\bar{f}$ at t = -1

$\bar{g}$ at f(-1) = 2 $\left(\frac{x-1}{x+1}\right)\frac{(x+1)-(x-1)}{(x+1)^2} = \frac{4(x-1)}{(x+1)^3} = \frac{-4}{1} = -4$

$\bar{f}$ at t = -1 = $t^{-2} - 1 = -2t^{-3} = -2 / t^3 = -2/-1 = 2$

$\overline{(gof)}$ at t = -1 = -4 * 2 = -8

Home Work : **1. solve above example by the second method ?**

2. prove that the following two curves intersect at the origin in right angle

$2y + 5x + x^4 - x^3 y^2 = 0$ & $5y - 2x + y^3 - x^2 y = 0$ (الحل في المحاضرات القديمة)

3. for the circle $x^2 + y^2 = r^2$ prove that $\frac{\bar{\bar{y}}}{(1+\bar{y}^2)^{\frac{3}{2}}} = -\frac{1}{r}$ (الحل في المحاضرات القديمة)

Example (16) : **If $\frac{d^2y}{dx^2} = t^2 + 1$, $\frac{dy}{dx} = t^3 + 3t$, find $\frac{dx}{dt}$?**

Solution:

$$\frac{d^2y}{dx^2} = \frac{\bar{y}/dt}{dx/dt}$$

$$\frac{\bar{y}}{dt} = 3t^2 + 3$$

$$\therefore \frac{dx}{dt} = \frac{\frac{\bar{y}}{dt}}{\frac{d^2y}{dx^2}} = \frac{3t^2+3}{t^2+1} = \frac{3(t^2+1)}{(t^2+1)} = 3$$

7.5. Derivatives of Trigonometric Functions :

1. $\frac{d}{dx}\sin x = \cos x$

2. $\frac{d}{dx}\cos x = -\sin x$

3. $\frac{d}{dx}\tan x = \sec^2 x$

4. $\frac{d}{dx} \sec x = \sec x \ \tan x$

5. $\frac{d}{dx} \csc x = -\csc x \ \cot x$

6. $\frac{d}{dx} \cot x = -\csc^2 x$

Example (17) : **Find $\frac{dy}{dx}$ if $xy + \sin y = 0$?**

Solution:

$$x \frac{dy}{dx} + y + \cos y \frac{dy}{dx} = 0$$

$$(x + \cos y) \frac{dy}{dx} + y = 0 \Longrightarrow \frac{dy}{dx} = \frac{-y}{x + \cos y}$$

Example (18) : **Find $\frac{dy}{dx}$ if $y = \cos^2 3x$?**

Solution:

$$\frac{dy}{dx} = 2 \cos 3x (-\sin 3x) * 3 = -6 \sin 3x \cos 3x$$

Home Work :

1. **Find $\frac{dy}{dx}$ if $y = \boldsymbol{tan} \sqrt{3x}$**
2. **Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{1+\cos 2x}{2}}$**
3. **Find $\frac{dy}{dx}$ if $y = \sec^2 5x$**

Example (19) : Find $\frac{d^2y}{dx^2}$ if $\frac{dy}{dx} = \sqrt{4 - \sin^2 t}$ and x = cos 2t ?

Solution:

$$\frac{d^2y}{dx^2} = \frac{\bar{y}/dt}{dx/dt}$$

$$\frac{dy}{dx} = \bar{y} = \sqrt{4 - \sin^2 t} \Longrightarrow \frac{\bar{y}}{dt} = \frac{1}{2}\,(4 - \sin^2 t)^{-\frac{1}{2}} * -2\sin t * \cos t$$

$$\frac{\bar{y}}{dt} = \frac{-2\sin t\cos t}{2\sqrt{4 - \sin^2 t}} = \frac{-\sin 2t}{2\sqrt{4 - \sin^2 t}}$$

$$\frac{dx}{dt} = -\sin 2t * 2 = -2\sin 2t$$

$$\frac{d^2y}{dx^2} = \frac{\bar{y}/dt}{dx/dt} = \frac{\frac{-\sin 2t}{2\sqrt{4-\sin^2 t}}}{-2\sin 2t} = \frac{\sin 2t}{4\sin 2t\sqrt{4-\sin^2 t}} = \frac{1}{4\sqrt{4-\sin^2 t}}$$

Example (20) : If $x^2y = 27$ when P (1, 3) and dy/dt = 10 , find the value of dx/dt at this time ?

Solution:

$$3x^2y^2\left(\frac{dy}{dx}\right) + 2xy^3 = 0$$

$$\frac{dy}{dx} = \frac{-2xy^3}{3x^2y^2} = -\frac{2y}{3x} = -2 \text{ (at point (1,3))}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ (chain rule)}$$

$$-2 = \frac{10}{dx/dt} \Longrightarrow dx/dt = -5$$

Example (21) : If $x = \sqrt{2t^2 + 1}$ and $y = (2t + 1)^2$ find dy/dx when t = 2 ?

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dt} = 2(2t + 1) * 2 = 4\,(2t + 1)$$

$$\frac{dx}{dt} = \frac{4t}{2\sqrt{2t^2 + 1}} = \frac{2t}{\sqrt{2t^2 + 1}}$$

$$\frac{dy}{dt})_{t=2} = \frac{2(4+1)\sqrt{8+1}}{2} = 15$$

Example (22) : **If $x = \cos^3 t$ and $y = \sin^3 t$ find dy/dx ?**

Solution:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = \cos^2 t + \sin^2 t$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

By implicit differentiation :

$$\frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}} * \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -(\frac{y}{x})^{1/3}$$

Another solution :

$$\frac{dy}{dt} = 3\sin^2 t \cos t$$

$$\frac{dx}{dt} = -3\cos^2 t \sin t$$

$$\frac{dy}{dx} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = \frac{-\sin t}{\cos t} = -(\frac{y}{x})^{1/3}$$

Example (23) : **If $x + \tan(xy) = 0$ find dy/dx ?**

Solution:

Using implicit differentiation :

$1 + \sec^2(xy)\,[\,x\,(dy/dx) + y\,] = 0$

$1 + \sec^2(xy) * x\,(dy/dx) + \sec^2(xy) * y = 0$

$$\frac{dy}{dx} = \frac{-[1 + y\sec^2(xy)]}{x\sec^2(xy)} = \frac{-1}{x\sec^2(xy)} - \frac{y}{x} = -\frac{1}{x}[y + \cos^2(xy)]$$

7.6. Derivatives of Inverse Function :

If $y = f(x)$, then , $x = f^{-1} y$

$F[f^{-1}(y)] = y$

$F^{-1}[f(x)] = x$

To find the inverse function :

1. Express x in term of y.
2. Change the position of x in y.

Example (23) : **Find the inverse function of $y = x^3 + 2$?**

Solution:

$x^3 = y - 2 = (y - 2)^{1/3}$

$y = (x - 2)^{1/3}$

7.6.1. The Inverse of Trigonometric Function :

1. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
4. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
5. $\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$

Example (24) : **Find dy/dx if $y = x\csc^{-1}(1/x)$?**

Solution:

$$\bar{y} = \csc^{-1}\frac{1}{x} + x\,\frac{-1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}\left(-\frac{1}{x^2}\right) = \csc^{-1}\frac{1}{x} + \frac{1}{x\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}$$

Example (25) : Find dy/dx if $\tan^{-1} y = x \sin y + x^3$?

Solution:

$$x\cos y\ \bar{y} + \sin y + 3x^2 = \frac{1}{1+y^2} * \bar{y}$$

$$\bar{y}\left(x\cos y - \frac{1}{1+y^2}\right) = -\sin y - 3x^2$$

$$\therefore\ \bar{y} = \frac{-\sin y - 3x^2 - y^2\sin y - 3x^2y^2}{x\cos y + xy^2\cos y - 1}$$

Example (26) : If $y = x\sin^{-1}x + \sqrt{1-x^2}$ show that $(1-x^2)\bar{\bar{y}} + x\,\bar{y} - y = 0$

Solution:

$$\bar{y} = x\,\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} * (-2x)$$

$$\bar{y} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

$$\bar{\bar{y}} = \frac{1}{\sqrt{1-x^2}}$$

$$(1-x^2) * \frac{1}{\sqrt{1-x^2}} + x\sin^{-1}x - x\sin^{-1}x - \sqrt{1-x^2}$$

$$\sqrt{1-x^2} * \sqrt{1-x^2} * \frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2} = 0$$

Example (27) : Find dy/dx for $\sin^{-1}(xy) = \cos^{-1}(x+y)$

Solution:

$$\frac{1}{\sqrt{1-(xy)^2}}\left(x\frac{dy}{dx}+y\right) = \frac{-1}{\sqrt{1-(x+y)^2}}\left(1+\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{\left(\frac{-1}{\sqrt{1-(x+y)^2}} - \frac{y}{\sqrt{1-(xy)^2}}\right)}{\left(\frac{x}{\sqrt{1-(xy)^2}} + \frac{1}{\sqrt{1-(x+y)^2}}\right)}$$

Example (28) : Find the maximum height of the curve y = 6 cos x – 8 sin x above the x – axis ?

Solution:

dy/dx = - 6 sin x – 8 cos x

- 6 sin x – 8 cos x = 0 ⟹ 6 sin x = - 8 cos x (÷ cos x)

6 tan x = -8

tan x = -8 / 6 = -4 / 3

x = $\tan^{-1}$ (-4 / 3)

Example (29) : Prove that $\tan^{-1}(-x) = -\tan^{-1}(x)$

Solution:

Let y = $\tan^{-1}$ (-x) ⟹ -x = tan y

x = - tan y ⟹ x = tan (-y)

- y = $\tan^{-1}$ x

y = - $\tan^{-1}$ x

∴ $\tan^{-1}(-x) = -\tan^{-1}(x)$

Example (30) : Find $\bar{y}$ for $y^2 \sin x + y = \tan^{-1}x$

Solution:

$$2y\bar{y}\sin x + y^2\cos x + \bar{y} = \frac{1}{1+x^2}$$

$$\bar{y}\,[\,(2y * \sin x) + 1\,] = -y^2\cos x + \frac{1}{1+x^2}$$

$$\bar{y} = \frac{\frac{1}{1+x^2} - y^2 \cos x}{[\,(2y \sin x)+1\,]}$$

Example (31) : Find $\bar{y}$ for $\frac{x^2 \sec^4\frac{y}{2} * \csc^4\frac{y}{2}}{\csc^4\frac{y}{2} + \sec^4\frac{y}{2}} = 1$

Solution:

$$x^2 \sec^4 \frac{y}{2} * \csc^4 \frac{y}{2} = \csc^4 \frac{y}{2} + \sec^4 \frac{y}{2} \quad (\div \csc^4 \frac{y}{2})$$

$$x^2 \sec^4 \frac{y}{2} = 1 + \tan^4 \frac{y}{2}$$

$$4\,x^2 \sec^3 \frac{y}{2} * \sec \frac{y}{2}\ \tan\frac{y}{2} * \frac{1}{2}\frac{dy}{dx} + 2x \sec^4\frac{y}{2} = 4\ \tan^3 \frac{y}{2} * \sec^2 \frac{y}{2} * \frac{1}{2}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2x^2 \sec^4 \frac{y}{2} * \tan\frac{y}{2} - 2\tan^3 \frac{y}{2} * \sec^2\frac{y}{2}\right) = -2x \sec^4 \frac{y}{2} \qquad (\div 2 \sec^2 \frac{y}{2})$$

$$\frac{dy}{dx}\left(x^2 \sec^2 \frac{y}{2} * \tan\frac{y}{2} - \tan^3 \frac{y}{2}\right) = -x \sec^4 \frac{y}{2}$$

$$\frac{dy}{dx} = \frac{x \sec^2 \frac{y}{2}}{-x^2 \sec^2 \frac{y}{2}\ \tan\frac{y}{2} + \tan^3\frac{y}{2}}$$

Example (32) : Prove that $\cos^{-1}(-x) + \cos^{-1}(x) = \pi$

Solution:

$y = \cos^{-1}(-x)$

$-x = \cos y$

$x = -\cos y = \cos(\pi - y)$

$(\pi - y) = \cos^{-1} x$

$\pi = \cos^{-1} x + y$

$\therefore \pi = \cos^{-1}(-x) + \cos^{-1}(x)$

7.7. Derivatives of Exponential & Logarithmic Functions :

1. $\frac{d}{dx} \ln x = \frac{1}{x}$

2. $\frac{d}{dx} e^x = e^x$

3. $\frac{d}{dx} e^{ax} = a\, e^{ax}$

4. $\frac{d}{dx} a^x = a^x \ln a$

5. $\frac{d}{dx} x^x = x^x (1 + \ln x)$

Example (33) : **Find $\frac{dy}{dx}$ if $y = \ln (x^2 + 1)$**

Solution:

Let $u = x^2 + 1 \Longrightarrow \frac{du}{dx} = 2x$

$y = \ln u \Longrightarrow \frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = \frac{1}{u} . 2x = \frac{2x}{1+ x^2}$$

Or : Applying rule 1 :

$$\frac{dy}{dx} = \frac{1}{1+x^2} * 2x = \frac{2x}{1+x^2}$$

Example (34) : **Find $\frac{dy}{dx}$ if $y = e^{x^2}$**

Solution:

$$\frac{dy}{dx} = 2\,x\,.\,e^{x^2}$$

Example (35) : **Find** $\frac{dy}{dx}$ **if** $y = 9^{x^2+5}$

Solution: $\frac{dy}{dx} = 9^{x^2+5} * \ln 9 * 2x = 2x \ln 9\,.\,9^{x^2+5}$

Example (36) : **Find** $\frac{dy}{dx}$ **if** $y = x^{x^2+1}$

Solution:

$$\frac{dy}{dx} = x^{x^2+1} * 2x\,(\,1 + \ln x\,)$$

Another solution :

$u = x^2 + 1 \Longrightarrow \frac{du}{dx} = 2x$ and $y = x^u \Longrightarrow \frac{dy}{du} = x^u(1 + \ln x\,)$

$$\frac{dy}{dx} = x^u(1 + \ln x\,)\,.\,2x = x^{x^2+1} * 2x\,(\,1 + \ln x\,)$$

Example (37) : **Find** $\frac{dy}{dx}$ **if** $y = (x^4 + 2)^5 + 5^{x^4+2}$

Solution:

$$\frac{dy}{dx} = 5(x^4 + 2)^4 * 4x^3 + 5^{x^4+2} \ln 5 * 4x^3 = 4x^3\,[\,5(x^4 + 2)^4 + 5^{x^4+2} \ln 5]$$

Example (38) : **Find** $\frac{dy}{dx}$ **if** $y = (x^2 + 1)^{\sin x}$

Solution:

$\ln y = \sin x\ \ \ln\ \ x^2 + 1$

$$\frac{1}{y}\,\frac{dy}{dx} = \sin x * \frac{2x}{x^2+1} + \ln(x^2 + 1) * \cos x$$

$$\frac{dy}{dx} = (x^2 + 1)^{\sin x}\,[\frac{2x \sin x}{x^2+1} + \cos x \ln(x^2 + 1)]$$

Example (39) : Find $\frac{dy}{dx}$ if $y = e^{\tan^{-1}x}$

Solution:

$$\frac{dy}{dx} = e^{\tan^{-1}x} * \frac{1}{x^2+1}$$

Example (40) : Find $\frac{dy}{dx}$ if $y = (x^2 + 1)^{\pi} + \pi^{\sin x}$

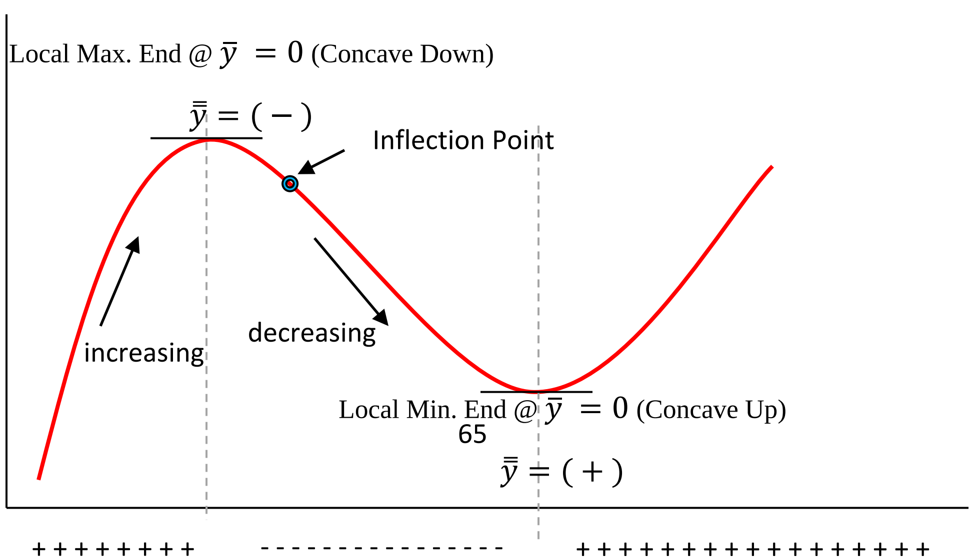
Solution:

$$\frac{dy}{dx} = \pi\,(x^2 + 1)^{\pi-1} * 2x + \pi^{\sin x} * \ln\pi * \cos x$$

Chapter Eight

APPLICATION OF DERIVATIVES

8.1. Application to Graphing :



Local max. $\Longrightarrow$ $\bar{y} = 0$

Local min. $\Longrightarrow$ $\bar{y} = 0$

1. If $\bar{y}$ transforms from positive to negative value, then the point is **Maximum**.
2. If $\bar{y}$ transforms from negative to positive value, then the point is **Minimum**.
3. **Concavity :**
   - If $\bar{y}$ decreases, then it's concave **Down**
   - If $\bar{y}$ increases, then it's concave **Up**

Or :

- $\bar{\bar{y}} < 0$ concave **Down**
- $\bar{\bar{y}} > 0$ concave **Up**

4. When the concavity changes, it is called a point of inflection point.

Example (1) : **Sketch the curve** $\mathbf{y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)}$

Solution:

1. Find y intercept when x = 0 $\Longrightarrow$ x = 0 & y = 1 (0,1)
2. Find $\bar{y}$:

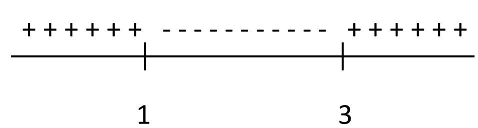
   $\bar{y} = \frac{1}{6}(3x^2 - 12x + 9)$

   $0 = \frac{1}{2}(x^2 - 4x + 3) = \frac{1}{2}(x - 3)(x - 1)$

x = 3 & x = 1

x = 1 the point is maximum

x = 3 the point is minimum



$\bar{\bar{y}} = \frac{1}{2}(x - 2)$

when x = 1 $\bar{\bar{y}}$ is negative (concave down)

when x = 3 $\bar{\bar{y}}$ is positive (concave up)

3. $\bar{\bar{y}} = 0$

$x-2 = 0$ $\quad\quad$ $x = 2$

when $x < 2$ $\quad\quad$ $\bar{\bar{y}}$ is negative

when $x > 2$ $\quad\quad$ $\bar{\bar{y}}$ is positive

∴ when $x = 2$ the point is **inflection**.

4. Conclusion :

x	y	$\bar{y}$	Sign of $\bar{\bar{y}}$	Remarks
-1	-5/3	4	-	Rising , concave down
0	1	3/2	-	Rising , concave down
1	5/3	0	-	Local max, concave down
2	4/3	-1/2	0	Falling, point of inflection
3	1	0	+	Local min, concave up
4	5/3	3/2	+	Rising, concave up

Home Work :

1. **Sketch** $y = \sin x$
2. **Sketch** $y = x^3 - 3x^2 + 4$
3. **Sketch** $y = x^3$
4. **Sketch** $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$
5. **Sketch** $y = x^{\frac{1}{3}}$
6. **Sketch** $y = x^4$

8.2. Application to Maximum & Minimum Points :

To solve the application on maximum and minimum point follow the following steps:

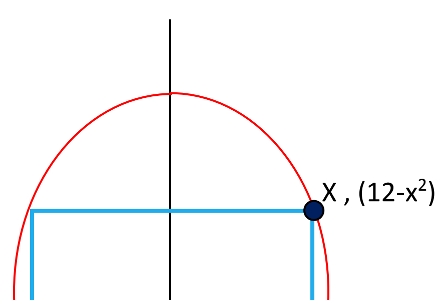
1. Draw a clear sketch of the problem.
2. Write a quantity which is want to find the maximum or minimum point in term of other quantities in this problem.
3. If the function or equation (step 2) contains more than one variable, we can use the condition in the problem to transform the equation depend on one variable.
4. Find the first derivative with respect to this variable.
5. Find the values of the variable which is give the zero of the first derivative.
6. Verify these values by using the second derivative to know the maximum or minimum point.

Example (2) : **Find the maximum area of a rectangle with lower base on the x – axis and upper base inside the parabola $y = 12 - x^2$?**

Solution:

A = base * height

$A = 2x * (12-x^2) = 24x-2x^3$



$\frac{dA}{dx} = 24 - 6\,x^2 \Longrightarrow 24 - 6\,x^2 = 0$

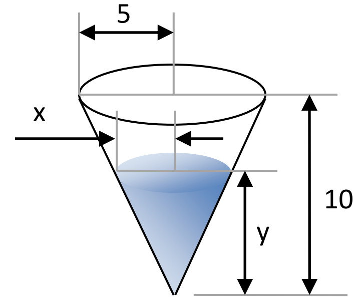
$x = \pm\, 2$

$\bar{\bar{y}} = -12\,x$

When $x = 2$, $\bar{\bar{y}} = -24$ (max. point)

$\therefore A = 4 * 8 = 32$ square unit.

Example (3) : **Water runs into a conical tank at the rate of 2 ft^3/min. , the tank stands point down and has a height of 10 ft and a base radius of 5 ft. , how fast is the water level rising when the water is 6 ft. deep ?**



Solution: $V = \frac{1}{3}\,\pi\, x^2 y$

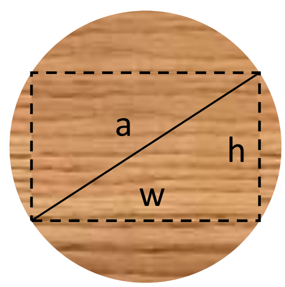
$\frac{y}{x} = \frac{10}{5} \Longrightarrow x = 0.5\,y$

$V = \frac{1}{12}\,\pi\, y^3$

$\frac{dV}{dt} = \frac{1}{12}\,\pi * 3 * y^2 * \frac{dy}{dt} \Longrightarrow 2 = \frac{1}{12}\,\pi * 3 * y^2 * \frac{dy}{dt} \Longrightarrow \frac{dy}{dt} = \frac{2}{9\pi} = 0.071$ ft./ min.

Example (4) : **A pier at a rectangular section is cut from a material at a circular section. If the strength of the pier is depend on the width and square height, find the dimension of the cross section of pier to give a maximum strength? (S = k.w.h^2)**

Solution:



$S = k\,.w\,.\,h^2$

$a^2 = h^2 + w^2 \Longrightarrow h^2 = a^2 - w^2$

$S = k\,.\,w\,(a^2 - w^2) = ka^2 w - kw^3$

$\frac{dS}{dw} = ka^2 - 3kw^2 \Longrightarrow ka^2 - 3kw^2 = 0$ $\quad (w = \frac{a}{\sqrt{3}})$

$ka^2 = 3kw^2 \Longrightarrow h^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3}$

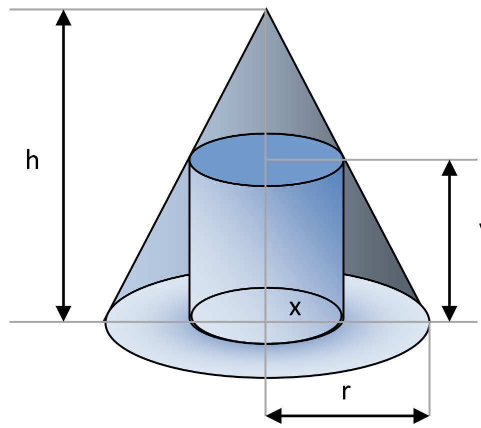
$h = \sqrt{2}\,w$

$\bar{\bar{S}} = -6\,kw = -6\,k\,\frac{a}{\sqrt{3}}$ $\quad \therefore$ the point is max. because the negative sign

---

Example (5) : **Find the volume of the maximum right circular cylinder that can be inscribed in a cone ? ($V = \pi x^2 y$)**

Solution:



$V = \pi x^2 y$

$\frac{r}{r-x} = \frac{h}{y} \Longrightarrow y = \frac{h(r-x)}{r}$

$V = \pi x^2 * \frac{h(r-x)}{r} = \pi x^2 * \frac{hr-hx}{r} = \pi x^2 h - \frac{\pi x^3 h}{r}$

$\frac{dV}{dx} = 2\pi hx - \frac{3\pi hx^2}{r}$

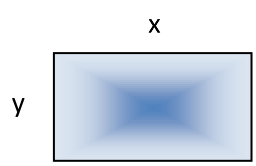
$\frac{2\pi rhx - 3\pi hx^2}{r} = 0 \Longrightarrow x = 2r/3$

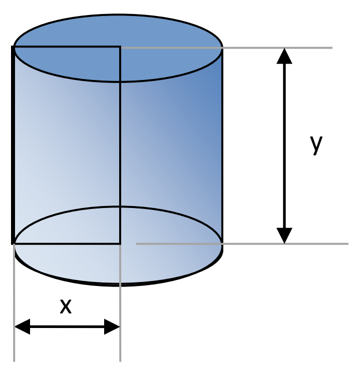
$\overline{\overline{V}} = 2\pi h - \frac{6\pi hx}{r}$ ∴ the point is max. because the negative sign

$$\therefore V = \pi \frac{4}{9} r^2 * \frac{h(r - \frac{2r}{3})}{r} = \frac{4\pi hr^2}{27}$$

Example (6) : **The rectangular sheet in the figure below is revolved about one of the edges of length y to sweep out the cylinder in figure shown. What values of x and y give the largest cylinder volume ? What is this volume ? ($x + y = 18$)**

Solution:





$V = \pi x^2 y = \pi x^2 (18 - x)$

$\frac{dV}{dx} = \pi (36x - 3x^2)$

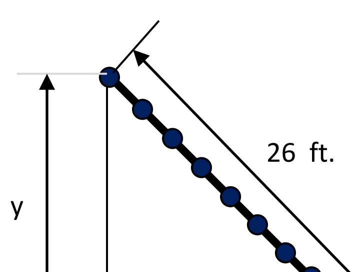
$\pi (36x - 3x^2) = 0$

$12x - x^2 = 0 \Longrightarrow x = 12$ & $y = 6$

$\therefore V = \pi * 12^2 * 6 = 2712.96$ square unit

Example (7) : **A ladder 26 ft. long rests on a horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 4 ft / sec. How fast is the top sliding down when the foot is 10 ft. from the wall ?**

Solution:



$x^2 + y^2 = (26)^2$

$\frac{dx}{dt} = 4$

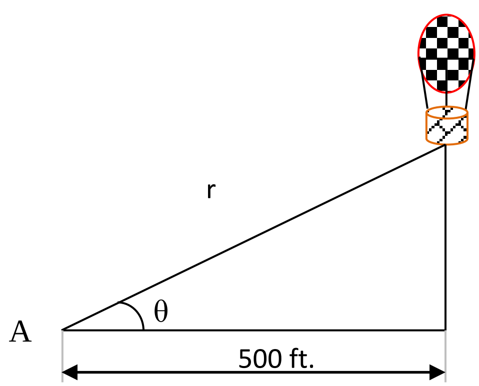
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longrightarrow \frac{dy}{dt} = \frac{-x}{y} * \frac{dx}{dt}$

$y = \sqrt{26^2 - 10^2} = 24$

$\therefore \frac{dy}{dt} = \frac{-10}{24} * 4 = \frac{-5}{3}$ ft / sec.

Example (8) : **A balloon rising from the ground at 140 ft / min. is attached a range finder at point A located 500 ft. from the point of lift off. Find the rate at which the angle at A and the ranger are changing when the balloon is 500 ft. above the ground ?**

Solution:



1. The angle at A :

$\tan\theta = y / 500$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{500} * \frac{dy}{dt}$ (when $y = 500$, $\theta = \pi / 4$)

$\sec^2\theta = (\sqrt{2})^2 = 2$ also $\frac{dy}{dt} = 140$ ft / min.

$2\frac{d\theta}{dt} = \frac{1}{500}(140) \Longrightarrow \frac{d\theta}{dt} = 0.14$ radian / min.

2. The range (r) $\Longrightarrow r^2 = 500 + y^2$

$2r\frac{dr}{dt} = 2y\frac{dy}{dt} \Longrightarrow \frac{dr}{dt} = \frac{y}{r}\frac{dy}{dt}$

When $y = 500$

$r = \sqrt{500^2 + 500^2} = 500\sqrt{2}$, $\frac{dy}{dt} = 140$ ft/ min.

$$\frac{dr}{dt} = \frac{500}{500\sqrt{2}} * 140 = 70\sqrt{2}$$

Chapter Nine

INTEGRATION

If the function f(x) is a derivative then the set of all anti derivatives of (f) is the indefinite integral of (f) with respect to x, denoted by **($\int f(x)dx$)**

The symbol $\int$ is an integral sign. We indicate this in integral notation by writing :

$$\int f(x)dx = f(x) + C$$

Where C is the constant of integration or arbitrary constant.

$f(x) = y = x^2$, $f(x) = x^2 + 1$, $f(x) = x^2 - 1$

$\bar{y} = 2x$, $\bar{y} = 2x$, $\bar{y} = 2x$

All functions with derivatives 2x on an interval is given that interval by the formula :

$f(x) = x^2 + C$

for example : evaluate $\int 2x\, dx = x^2 + C$

Example (1) : **Evaluate** $\int \frac{1}{\sqrt{x}}\, dx$

Solution:

$$\int \frac{1}{\sqrt{x}}\, dx = \int x^{\frac{-1}{2}}\, dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{x} + C$$

9.1. *Rules for Indefinite Integration :*

Rule (1) : *Constant Multiple Rule :*

$$\int K\, f(x) dx = K \int f(x) dx$$

Rule (2) : *Negative Sign Rule :*

$$\int - f(x) dx = - \int f(x) dx$$

Rule (1) : *Sum & Difference Rule :*

$$\int [\, f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Example (2) : **Evaluate** $\int (\, x^2 + 2x + 5)\, dx$

Solution:

$$\int x^2dx + \int 2x\,dx + \int 5\,dx = \frac{x^3}{3} + x^2 + 5x + C$$

9.2. Differential Equations :

y = f (x) is the solution of differential equation ($\frac{dy}{dx} = f(x)$)

for example :

$\frac{dy}{dx} = 2xy^2$ (differential equation from the first order)

$\frac{d^2y}{dx^2} + 6xy\frac{dy}{dx} + 3x^2y^3 = 0$ (differential equation from the second order)

Example (3) : **Solve the differential equation $\frac{dy}{dx} = 3x^2$**

Solution: $\frac{dy}{dx} = 3x^2 \Longrightarrow dy = 3x^2\,dx$

$$\int dy = \int 3x^2dx$$

$y + c_1 = x^3 + C_2 \Longrightarrow y = x^3 + C$ where ($C_1 + C_2 = C$)

Example (4) : **Solve the differential equation $\frac{dy}{dx} = x^2\sqrt{y}$**

Solution:

$$\frac{dy}{\sqrt{y}} = x^2dx$$

$$\int \frac{dy}{\sqrt{y}} = \int x^2dx$$

$$2\sqrt{y} = \frac{x^3}{3} + C$$

Example (5) : **Solve the differential equation $\frac{dy}{dx} = (x^2 + 5)^2$**

Solution:

$$\frac{dy}{dx} = (x^2 + 5)^2$$

$$\int dy = \int (x^2 + 5)^2 \, dx$$

$$\int dy = \int (x^4 + 10\,x^2 + 25) \, dx$$

$$y = \frac{x^5}{5} + \frac{10x^3}{3} + 25\,x + C$$

Example (5) : **Evaluate $\int (x^2 + 5)^2 * 2x\, dx$**

Solution: let $u = x^2 + 5$, $du = 2x\, dx$

$$\int u^2 du = \frac{u^3}{3} + C = \frac{(x^2 + 5)^3}{3} + C$$

Example (6) : **Evaluate $\int \sqrt{2x + 1}\, dx$**

Solution: let $u = 2x + 1$, $du = 2\, dx$

$$\frac{1}{2}\int u^{\frac{1}{2}}\, du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{(2x + 1)^{\frac{3}{2}}}{3} + C$$

Home Work :

1. **Evaluate $\int x\,(x + 1)^2\, dx$**
2. **Evaluate $\int \frac{x\, dx}{1+x}$**
3. **Evaluate $\int \frac{x}{\sqrt{x^2 - 1}}\, dx$**
4. **Evaluate $\int (2x^3 + 1)\, x^2\, dx$**
5. **Solve the differential equation $\frac{dy}{dx} = x + \sqrt{2x}$**
6. **Solve the differential equation $\frac{dy}{dx} = \frac{\sqrt{x+1}}{\sqrt{y-1}}$**
7. **Solve the differential equation $\frac{dy}{dx} = \sqrt{x\, y}$**
8. **The graph y = f(x) passes through the point (9,4) also the line tangent to the graph at any point (x , y) has the slope $3\sqrt{x}$ find f(x) ?**

Example (7) : **Find the curve whose slope at the point (x , y) is $3x^2$ if the curve is also passes through the point (1 , -1) ?**

Solution:

$\frac{dy}{dx} = 3\,x^2 \Longrightarrow \int dy = \int 3\,x^2\,dx$

$y = x^3 + C \Longrightarrow -1 = 1 + C \Longrightarrow C = -2$

$y = x^3 - 2$

Note :

$\text{acceleration (a)} = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, $\text{Velocity (v)} = \frac{ds}{dt}$

Example (8) : **A particle moves with acceleration $a = \sqrt{t} - \frac{1}{\sqrt{t}}$, assuming that the velocity V = 2 and the distance S = 5 when t = 0 , find :**

a. **The velocity V in terms of t.**
b. **The distance S in terms of t.**

Solution:

a. $\frac{dv}{dt} = a = \sqrt{t} - \frac{1}{\sqrt{t}}$

$\int dv = a = \int (\sqrt{t} - \frac{1}{\sqrt{t}})\ dt$

$V = \frac{2}{3}\,t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + C \Longrightarrow 2 = 0 - 0 + C \Longrightarrow C = 2$

$V = \frac{2}{3}\,t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + 2$

b. $\frac{ds}{dt} = V = \frac{2}{3}\,t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + 2$

$\int ds = \int \frac{2}{3}\,t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + 2\ \ dt$

$S = \frac{4}{15}\,t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + 2t + C \Longrightarrow 5 = 0 - 0 + 0 + C \Longrightarrow C = 5$

$S = \frac{4}{15} t^{\frac{5}{2}} - \frac{4}{3} t^{\frac{3}{2}} + 2t + 5$

Example (9) : **Evaluate $\int \frac{x}{\sqrt{4-x^2}} \, dx$**

Solution:

$$\int x\,(4 - x^2)^{\frac{-1}{2}} \, dx$$

Let $u = 4 - x^2$, $du = -2x\,dx$ $x\,dx = -\,du/2$

$$\frac{-1}{2} \int u^{\frac{-1}{2}} \, du = \frac{-1}{2} * \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = -\,u^{\frac{1}{2}} + C = -(4 - x^2)^{\frac{1}{2}} + C$$

Example (10) : **Evaluate $\int \frac{(z+1)}{\sqrt[3]{3z^2+6z+5}} \, dz$**

Solution:

Let $u = 3z^2 + 6z + 5$

$6\,dz\,(z + 1) = du \implies \frac{du}{6} = dz\,(z + 1)$

$$\frac{1}{6}\int \frac{du}{u^{1/3}} = \frac{1}{6}\int u^{-1/3}\,du = \frac{1}{6} * \frac{3}{2} * u^{2/3} + C = \frac{1}{4} u^{2/3} + C = \frac{1}{4} \sqrt[3]{(3z^2 + 6z + 5)^2} + C$$

Example (11) : **Evaluate $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2} \, dx$**

Solution:

Let $u = 1 + \sqrt{x} \implies du = \frac{1}{2\sqrt{x}} dx \implies 2du = \frac{1}{\sqrt{x}} dx$

$= \int \frac{2\,du}{u^2} = \frac{-2}{u} + C = \frac{-1}{1+\sqrt{x}} + C$

Example (12) : **Solve the differential equation $\frac{dy}{dx} = \frac{4\sqrt{(1+y^2)^3}}{y}$ when x = 0 & y = 0**

Solution:

$$\int dx = \int \frac{y\,dy}{4\,(1+y^2)^{3/2}}$$

Let $u = (1 + y^2)$ & $du = 2y\,dy$

$$= \frac{1}{8}\int u^{-3/2}\ du = \frac{-1}{4}\ u^{-1/2} + C \qquad \frac{-1}{4\sqrt{1+y^2}} + C = x$$

$$\frac{-1}{4} + C = 0 \Longrightarrow C = \frac{1}{4}$$

$$\frac{-1}{4\sqrt{1+y^2}} + \frac{1}{4} = x$$

$$4\,x + \frac{1}{\sqrt{1+y^2}} - 1 = 0$$

Example (13) : **Solve the differential equation $\frac{dr}{dz} = 24\,z\,(3z^2 - 1)^3$ r = -3 when z = 0**

Solution:

$$\int dr = \int 24\,z\,(3z^2 - 1)^3\,dz$$

Let $u = 3z^2 - 1$ & $du = 6z\,dz \Longrightarrow 24\,z\,dz = 4\,du$

$$= \int 4\,u^3\,du = u^4 + C$$

$$r = (3z^2 - 1)^4 + C$$

$C = -4$

$$r = (3z^2 - 1)^4 - 4$$

Example (14) : **Solve the differential equation $2y\frac{dy}{dx} = 3\,x\sqrt{x^2+1}\ \sqrt{y^2+1}$ y = 0 when x = 0**

Solution:

$$\frac{2y\,dy}{\sqrt{y^2+1}} = \int 3x\sqrt{x^2+1}\ dx$$

Let $u = (y^2 + 1) \Longrightarrow du = 2y\,dy$

$\Longrightarrow$ $\Longrightarrow$

Let v = $(x^2 + 1)$ dv = 2x dx $\frac{3}{2}$ dv = 3x dx

$\int u^{-1/2}\ du = \frac{3}{2}\int v^{\frac{1}{2}}\ dv$

$2\ u^{1/2} = v^{3/2} + C$

$2\sqrt{y^2 + 1} = (x^2 + 1)^{3/2} + C$

x = 0 , y = 0

2 = 1 + C C = 1

$2\sqrt{y^2 + 1} = (x^2 + 1)^{3/2} + 1$

9.3. *Integral of Trigonometric Functions :*

Example (15) : **Evaluate $\int \frac{\cos 2x}{\sin^3 2x}\ dx$** **(assume the value that have largest power U)**

Solution:

Let u = sin 2x , du = 2 cos 2x dx $\Longrightarrow$ $\frac{du}{2} = \cos 2x\ dx$

$= \frac{1}{2}\int \frac{du}{u^3} = \frac{1}{2}\frac{u^{-2}}{-2} = \frac{-1}{4\ u^2} + C = \frac{-1}{4\ sin^2\ 2x} + C$

Example (16) : **Evaluate $\int 16x \sin^3(2x^2 + 1) \cos(2x^2 + 1)\ dx$**

Solution:

Let u = $\sin(2x^2 + 1)$, du = $4x \cos(2x^2 + 1)$ dx $\Longrightarrow$ $4\ du = 16\ x \cos(2x^2 + 1)\ dx$

$\int 4\ u^3 du = u^4 + C = \sin^4(2x^2 + 1) + C$

Example (17) : **Evaluate $\int \frac{1}{\cos^2 2x}\ dx$**

Solution:

$\int \frac{1}{\cos^2 2x}\ dx = \int \sec^2 2x\ dx$

Let u = 2x $\Longrightarrow$ du = 2 dx $\Longrightarrow$ $\frac{du}{2} = dx$

$$\frac{1}{2}\int \sec^2 u\, du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2x + C$$

Note :

1. If we don't find the derivative of the function, we use the geometrical identities by substituting them in the function and finding the integration.
2. If the power is even (for example $\sin^2 x$) we always apply the law of double angle (i.e. $\frac{1-\cos 2x}{2} = \sin^2 x$ or $1 + \tan^2 x = \sec^2 x$) .
3. If the power is odd, we divide this power into two parts even and odd and substituting an identity instead of the function that have an even power .

Example (18) : Evaluate $\int \sin^2 x\, dx$

Solution:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example (19) : Evaluate $\int \sin^3 x\, dx$

Solution:

$$\int \sin x \sin^2 x\, dx = \int \sin x\,(1 - \cos^2 x)dx = \int \sin x\, dx - \int \sin x \cos^2 x\, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

Example (20) : Evaluate $\int \tan^2 x\, dx$

Solution:

$$\int (\sec^2 x - 1)dx = \int \sec^2 x\, dx - \int dx = \tan x - x + C$$

Example (21) : **Evaluate $\int \frac{7\tan^6 x}{\cos^2 x} dx$**

Solution:

$$\int \frac{7\tan^6 x}{\cos^2 x} dx = \int 7 \tan^6 x \sec^2 x\, dx = \frac{7\tan^7 x}{7} + C = \tan^7 x + C$$

Example (22) : **Evaluate $\int \frac{\sin^2 x}{1+\cos 2x} dx$**

Solution:

$$\int \frac{\sin^2 x}{1+\cos 2x} dx = \int \frac{(1-\cos 2x)(1+\cos 2x)}{(1+\cos 2x)} dx = x - \frac{1}{2}\sin 2x + C$$

Example (22) : **Evaluate $\int \cos^4 2x \sin^3 2x\, dx$** **(separate the smaller integer power to get the odd power of sin)**

Solution:

$$\int \cos^4 2x \sin^3 2x\, dx = \int \cos^4 2x \sin^2 2x \sin 2x\, dx = \int \cos^4 2x\,(1 - \cos^2 2x) \sin 2x\, dx$$

$$= \int \cos^4 2x \sin 2x\, dx - \int \cos^6 2x \sin 2x\, dx = \frac{-1}{2}\frac{\cos^5 2x}{5} + \frac{1}{2}\frac{\cos^7 2x}{7} + C$$

Example (23) : **Evaluate $\int \tan^3 3x \sec^4 3x\, dx$** **(separate the power of sec to get the derivative of tan)**

Solution:

$$\int \tan^3 3x \sec^4 3x\, dx = \int \tan^3 3x \sec^2 3x \sec^2 3x\, dx = \int \tan^3 3x\,(1 + \tan^2 3x) \sec^2 3x\, dx$$

$$= \int \tan^3 3x \sec^2 3x\, dx + \int \tan^5 \sec^2 3x\, dx = \frac{\tan^4 3x}{12} + \frac{\tan^6 3x}{18} + C$$

Example (24) : **Evaluate $\int \sec x\, dx$**

Solution:

$$\int \sec x\, dx = \int \frac{1}{\cos x} \cdot \frac{\cos x}{\cos x} dx = \int \frac{\cos x}{1-\sin^2 x} dx$$

Let $u = \sin x$ & $du = \cos x\, dx$

$$\int \frac{du}{1-u^2} = \tan^{-1} u + C = \tan^{-1}(\sin x) + C$$

Example (25) : **Evaluate $\int \cos^{2/3} x \ \sin^5 x\, dx$** **(the power of sin is integer , odd & positive (separate))**

Solution:

$\int \cos^{2/3} x \;\; \sin^5 x \;\; dx = \int \cos^{2/3} x \;\; \sin^4 x \sin x \;\; dx = \int \cos^{2/3} x \;\; (1-\cos^2 x)^2 \;\; \sin x \;\; dx$

Let u = cos x & du = - sin x dx

$= -\int u^{2/3} (1-u^2)^2 \;\; dx = -\int u^{2/3} (1 - 2u^2 + u^4) du = -\int \left(u^{2/3} - 2u^{8/3} + u^{14/3} \right) du$

$= -\cos^{5/3} x \left(\frac{3}{5} - \frac{6}{4} \cos^2 x + \frac{3}{17} \cos^4 x \right) + C$

Example (26) : **Evaluate $\int \cos^4 x \;\; dx$**

Solution:

$$\int \cos^4 x \;\; dx = \int \cos^2 x \;\; \cos^2 x \;\; dx = \int \frac{(1+\cos 2x)}{2} \cdot \frac{(1+\cos 2x)}{2} \, dx = \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$\frac{1}{4} \int (1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x) \, dx = \frac{1}{4} (x + \sin 2x + \frac{1}{2} x + \frac{1}{8} \sin 4x + C$$

Home Work :

1. **Evaluate $\int \tan^3 x \sec^2 x \, dx$**
2. **Evaluate $\int \tan x \sec^3 x \, dx$**
3. **Evaluate $\int \tan x \sec^{3/2} x \, dx$**
4. **Evaluate $\int \frac{\sin 2t}{\sqrt{2-\cos 2t}} dt$**
5. **Evaluate $\int \cos^3 dx$**
6. **Evaluate $\int \tan x \sec^2 x \, dx$**
7. **Evaluate $\int 5 \tan x \sec x \, dx$**
8. **Evaluate $\int \frac{dx}{\sin x \cos x}$**

9.3. Integral of the Natural Logarithmic :

$\int \frac{1}{u} \, du = \ln |u| + C$ **(absolute sign because (ln) always positive)**

Example (27) : **Evaluate $\int \frac{\cos \theta}{2+\sin \theta} d\theta$**

Solution: Let u = 2 + sin θ (usually, we assume the term which contains a number (u))

$du = \cos \theta \, d\theta$

$$\int \frac{du}{u} = \ln|u| + C = \ln \; |2 + \sin \theta| + C$$

Example (28) : **Evaluate $\int \sec x \, dx$**

Solution: $\int \sec x * \frac{\sec x+\tan x}{\sec x+\tan x} dx = \int \frac{sec^2 x+\sec x \tan x}{\sec x+\tan x} dx$

Let u = sec x + tan x , du = (sec x tan x + $\sec^2$ x) dx

$$\int \frac{du}{u} = \ln|u| + C = \ln |\sec x + \tan x| + C$$

Example (29) : **Evaluate $\int 2x \tan (5 x^2 - 1) \, dx$**

Solution: Let u = 5 x^2 – 1 ⟹ du = 10 x dx ⟹ $\frac{du}{5} = 2x\,dx$

$$= \frac{1}{5} \ln |\sec u| + C = \frac{1}{5} \ln |\sec (5x^2 - 1)| + C$$

Home Work :

1. **Prove $\int \tan u \, du = -\ln |\cos u| + C = \ln |\sec u| + C$**
2. **Prove $\int \cot u \, du = \ln |\sin u| + C$**

Example (30) : **Evaluate $\int \frac{1}{x (\ln x)^2} dx$**

Solution:

Let u = ln x ⟹ du = $\frac{1}{x}$ dx

$$\int \frac{du}{u^2} = \int u^{-2} \, du = \frac{-1}{u} + C = \frac{-1}{\ln x} + C$$

Example (31) : **Evaluate $\int \frac{1}{(1+x^2)\tan^{-1}x} dx$**

Solution:

Let u = $\tan^{-1}$ x ⟹ du = $\frac{1}{1+x^2}$ dx

$$\frac{du}{u} = \ln|u| + C = \ln|\tan^{-1}x| + C$$

Example (32) : **Evaluate** $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})} dx$

Solution:

Let $u = 1 + \sqrt{x} \Longrightarrow 2du = \frac{1}{\sqrt{x}} dx$

$$\int \frac{2\,du}{u} = 2\ln u + C = 2\ln(1+\sqrt{x}) + C$$

Example (33) : **Evaluate** $\int \frac{2\cos x}{(1+\sin^2 x)} dx$ **(about inverse function)**

Solution: Let $u = \sin x$ $\quad$ $2du = 2\cos x\, dx$

$= 2\int \frac{du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}(\sin x) + C$

9.4. Integral of the Exponential Functions :

Notes : (simplifying the form of the function)

1. $\ln e^x = x$ $\quad$ ($\ln e = 1$)
2. $e^{\ln x} = x$
3. $a^{\log x} = \frac{\ln x}{\ln a}$

for example : 1. $e^{\ln(x^2+1)} = (x^2+1)$

3. $\ln \frac{e^{2x}}{5} = \ln e^{2x} - \ln 5 = 2x - \ln 5$

Home Work : simplify :

1. $e^{\ln x - 2\ln y}$
2. $\ln x^2 . e^{-2x}$
3. $e^{x + \ln x}$

Solve for y :

1. $\ln(y-2) = \ln(\sin x) - x$
2. $e^y = e^{x^2} . e^{2x+1}$
3. $\ln(y^2-1) - \ln(y+1) = \sin x$

Example (34) : **Solve for y** **1. $\ln y = x^2$** **2. $e^{3y} = 2 + \cos x$** **3. $\ln(y-1) - \ln y = 3x$**

Solution:

1. $e^{\ln y} = e^{x^2} \Longrightarrow y = e^{x^2}$
2. $e^{3y} = 2 + \cos x \Longrightarrow \ln e^{3y} = \ln 2 + \ln \cos x \Longrightarrow y = \frac{1}{3}(\ln 2 + \ln \cos x)$
3. $\ln(y-1) - \ln y = 3x \Longrightarrow \ln\frac{(y-1)}{y} = 3x \Longrightarrow e^{\ln\frac{(y-1)}{y}} = e^{3x} \Longrightarrow y = \frac{1}{(1-e^{3x})}$

Example (35) : **Evaluate $\int \frac{e^{\tan^{-1} x}}{1+x^2}$**

Solution: Let u = $\tan^{-1} x$ $\quad du = \frac{1}{1+x^2}\,dx$

$$\int e^u\,du = e^u + C = e^{\tan^{-1}x} + C$$

Example (36) : **solve the differential equation (solve for y) $\frac{dy}{dx} = 2xe^{-y}$ when x = 2 & y = 0**

Solution:

$\frac{dy}{dx} = 2xe^{-y} \Longrightarrow \int e^y dy = \int 2x\,dx \Longrightarrow e^y = x^2 + C \Longrightarrow 1 = 4 + C \Longrightarrow C = -3$

$\ln e^y = \ln x^2 - \ln 3 \qquad y = \ln(x^2 - 3)$

Example (36) : **show that $y = c\,e^{ax}$ is a solution of the differential equation $\frac{dy}{dx} = a\,y$, where c constant**

Solution:

$$\int \frac{dy}{ay} = \int dx$$

$\frac{1}{a}\ln y = x + C \Longrightarrow \ln y = ax + C$

$e^{\ln y} = e^{ax} + e^c \Longrightarrow y = C\,e^{ax}$

Example (37) : **Evaluate $\int \frac{e^x}{4+9e^{2x}}\,dx$**

Solution:

$\int \frac{e^x}{4+9e^{2x}}dx = \int \frac{e^x}{4(1+(\frac{3}{2}e^x)^2}\ dx$

Let $u = \frac{3}{2}\ e^x$ $\quad du = \frac{3}{2}\ e^x\ dx$ $\quad \frac{2}{3}\ du = e^x\ dx$

$\frac{1}{6}\int \frac{du}{1+u^2} = \frac{1}{6}\ \tan^{-1}\left(\frac{3}{2}\ e^x\right) + C$

Example (38) : **Evaluate $\int \frac{dx}{e^x+e^{-x}}$**

Solution:

divide the numerator & denominator by e^{-x}

$\int \frac{e^x\ dx}{e^{2x}+1}$

Let $u = e^x$ & $du = e^x\ dx$

$$\int \frac{du}{u^2+1} = \tan^{-1}u + C = \tan^{-1} e^x + C$$

Home Work :

1. **Find the value of the constant (r) for which $y = e^{rx}$ is a solution of the differential equation**

   $\ddot{y} - 4\dot{y} + 4y = 0$ (Ans : r = 2)

2. **Solve the differential equation $e^y\frac{dy}{dx} = 2x\ e^{x^2-1}$ $\quad y = 0$ & $x = 1$ (Ans : $y = x^2 - 1$)**
3. **Solve the differential equation $\frac{1}{y+1}\ \frac{dy}{dx} = \frac{1}{2x}$**
4. **Evaluate $\int \frac{6e^{1/x}}{x^2}\ dx$ (Ans : $-6e^{1/x} + C$)**

9.5. The Function (a^x) :

$\int a^x\ dx = \frac{a^x}{\ln a} + C$, $\quad a^x = (e^{\ln a})^x = e^{x\ln a}$

Example (39) : **Simplify (1) $e^{3\ln x}$ (2) $e^{3\ln 2}$**

Solution: 1. $e^{3\ln x} = e^{\ln x^3} = x^3$ $\qquad$ 2. $e^{3\ln 2} = 2^3 = 8$

Example (40) : **Evaluate $\int 2^x\ dx$**

Solution: $\int 2^x\ dx = \frac{2^x}{\ln 2} + C$

Example (41) : **Evaluate $\int 2^{\sin x} \cos x \, dx$**

Solution: Let u = sin x & du = cos x dx $\Longrightarrow \int 2^u \, du = \frac{2^u}{\ln 2} + C = \frac{2^{\sin x}}{\ln 2} + C$

Example (42) : **Evaluate $\int 3^{2x} \, dx$**

Solution: $\int 3^{2x} \, dx = \frac{3^{2x}}{2 \ln 3} + C$

Home Work :

1. $\int (\frac{1}{10})^x \, dx$ (Ans : $\frac{10}{\ln 10} - \frac{1}{10 \ln 10}$)
2. $\int x \, . \, 2^{-x^2} \, dx$ (Ans : $\frac{1}{8 \ln 2}$)
3. $\int 2^{\cos x} \sin x \, dx$ (Ans : $\frac{1}{\ln 2}$)
4. $\int 2^{\sec x} \sec x \tan x \, dx$ (Ans : $\frac{2}{\ln 2}$)

9.6. Integral of Hyperbolic Functions :

Example (43) : **Evaluate $\int \operatorname{sech}^2(2x-1) \, dx$**

Solution:

Let u = 2x-1 & du = 2 dx $\Longrightarrow \frac{1}{2} du = dx$

$$= \frac{1}{2} \int \operatorname{sech}^2 u \, du = \frac{1}{2} \tanh (2x-1) + C$$

Example (44) : **Evaluate $\int \frac{dx}{9x^2-25}$**

Solution:

$$\int \frac{dx}{9x^2 - 25} = \frac{1}{25} \int \frac{dx}{\frac{9}{25} x^2 - 1} = \frac{1}{25} \int \frac{dx}{(\frac{3}{5} x)^2 - 1}$$

Let u = $\frac{3}{5}$ x $\Longrightarrow$ du = $\frac{3}{5}$ dx $\Longrightarrow \frac{5}{3}$ du = dx

$$\frac{1}{25} * \frac{5}{3} \int \frac{du}{u^2 - 1} = \frac{-1}{15} \coth^{-1} u + C = \frac{-1}{15} \coth^{-1} (\frac{3}{5} x) + C$$

Example (45) : Evaluate $\int 4\cosh^2 t \; dt$

Solution:

$$\int 4\cosh^2 t \; dt = \int 4\,\frac{(1+\cosh 2t)}{2}\; dt = \int 2\,(1+\cosh 2t)\; dt = 2t + \sinh 2t + C$$

Home Work :

1. $\int \tanh^4 x \; dx$
2. $\int \sinh^3 3x \cosh^5 3x \; dx$
3. $\int \frac{\sinh(\frac{z-1}{3})}{\cosh(\frac{z-1}{3})} \; dx$

Chapter Ten

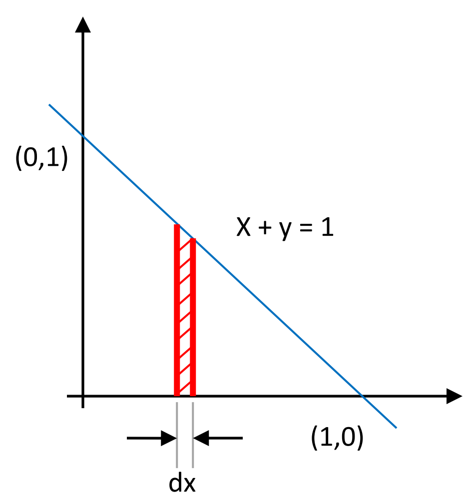
APPLICATION ON DEFINITE INTEGRATION

(AREA UNDER CURVE)

Example (1) : Find the area between the line $x + y = 1$ & $x-$axis ?

Solution:

$$A = \int_0^1 (1-x)dx = x - \frac{x^2}{2}\;\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2} \quad \text{square unit}$$

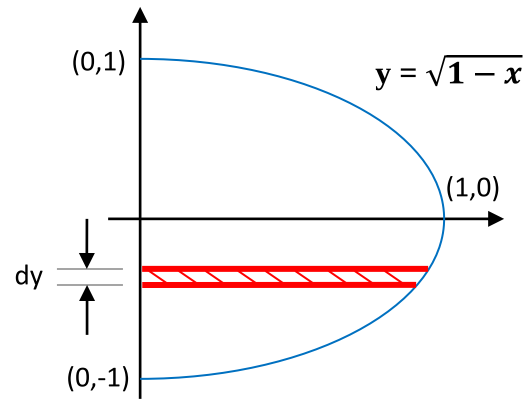


Example (2) : Find the area between the curve $y = \sqrt{1-x}$ & y – axis ?

Solution:

$x = 1 - y^2$

$$A = \int_{-1}^{1} 1 - y^2 \, dy = y - \frac{y^3}{3} \Big|_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \ \textit{square unit}$$



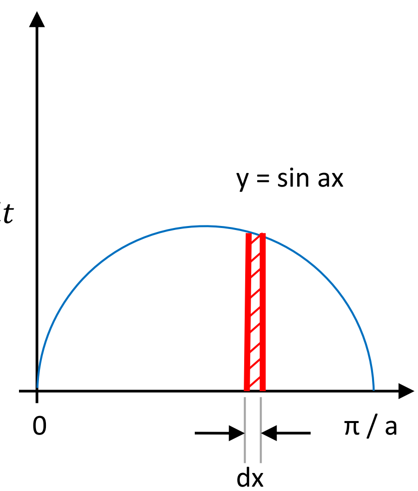
Example (3) : Find the area between the curve y = sin ax & x – axis ?

Solution:

if $ax = 0 \Longrightarrow x = 0$

if $ax = \pi \Longrightarrow x = \pi / a$

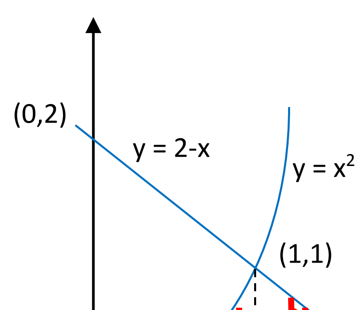
$$A = \int_0^{\pi/a} \sin ax \, dx = \frac{-1}{a} \cos ax \Big|_0^{\pi/a} = \frac{-1}{a} [-1 - 1] = \frac{2}{a} \ \textit{square unit}$$



Example (4) : Find the area enclosed between the curve $y = x^2$ & the line y = 2 – x and the x – axis ?

Solution:

$x^2 = 2 - x$



(x+2) (x-1) = 0

x = -2 & x = 1

$$A = \int_0^1 x^2\,dx + \int_1^2 (2-x)dx = \frac{x^3}{3}\Big|_0^1 + \left[2x - \frac{x^2}{2}\right]\Big|_1^2 = \frac{5}{6}\ \textit{square unit}$$

Example (5) : Find the area enclosed between the curve y = sin x & the line y = cos x between x = 0 and x = π / 2 ?

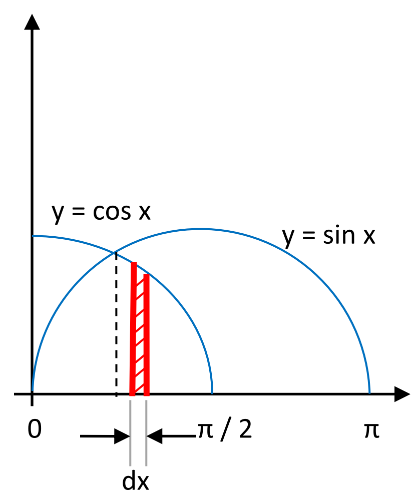
Solution:

sin x = cos x

tan x = 1 ⟹ x = 45^0

$$A = \int_0^{\pi/4} \sin x\ dx - \int_{\pi/4}^{\pi/2} \cos x\,dx = -\cos x\,\Big|_0^{\pi/4} + \sin x\,\Big|_{\pi/4}^{\pi/2}$$

$$\left(\frac{-1}{2} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \frac{2}{\sqrt{2}} = 0.5858$$



Example (6) : Find the area enclosed between the curve y = x^2 & the line y = 3x and for x = 1 and x=2?

Solution:

x^2 - 3x = 0 ⟹ x (x – 3) = 0



x = 0 and x = 3 (original integration limits)

A = $\int_1^2 (3x - x^2)dx = \frac{3x^2}{2} - \frac{x^3}{3}\Big|_1^2 = \frac{13}{6}$ *square unit*

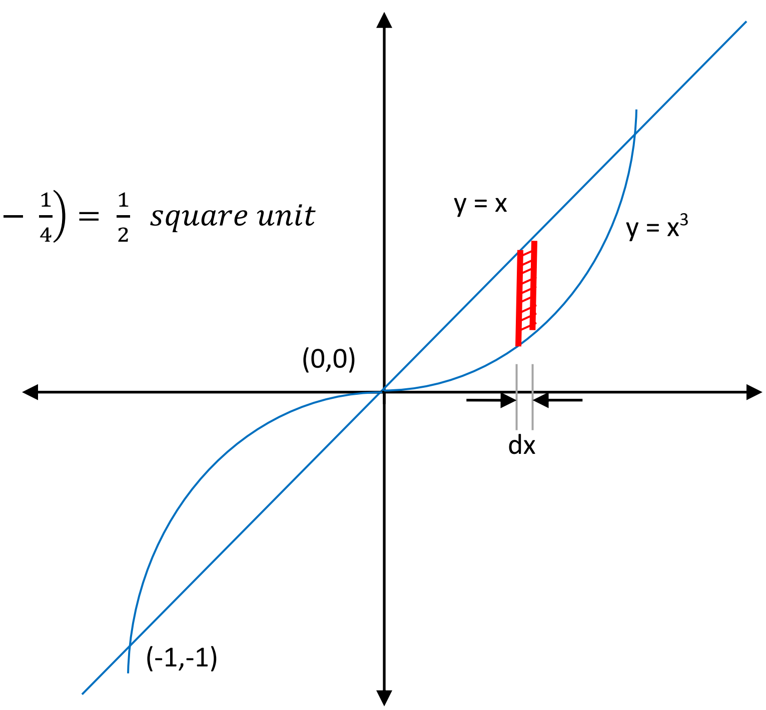
Example (7) : Find the area which is bounded by $y = x^3$ & the line $y = x$?

Solution:

$x^3 - x = 0 \Longrightarrow x(x^2 - 1) = 0$

x = 0 & x = ± 1 & y = ± 1

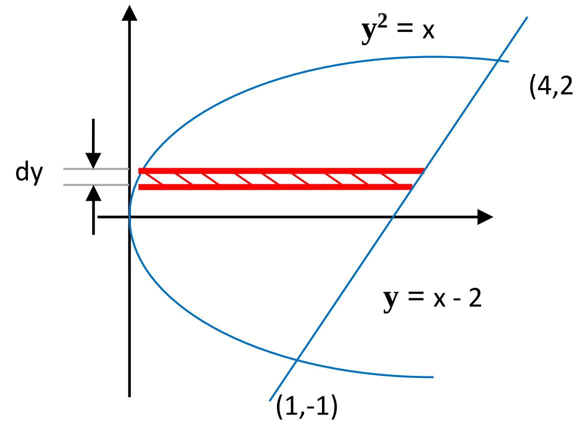
A = $2\int_0^1 (x - x^3)dx = 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$ *square unit*



Example (8) : Find the area which is bounded by $y^2 = x$ & the line $y = x - 2$?

Solution:

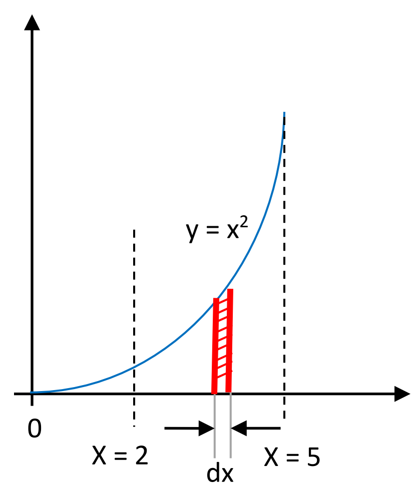
$A = \int_{-1}^{2} (y + 2 - y^2) dy = \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_{-1}^{2} = \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 4.5$



Example (9) : Find the area which is bounded by $y = x^2$ & the line $y = 0$ at $x = 2$ to $x = 5$?

Solution:

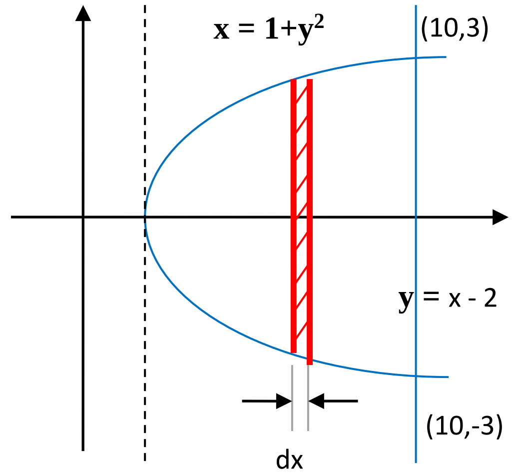
$A = \int_{2}^{5} (x^2) dx = \frac{x^3}{3} \Big|_{2}^{5} = \left(\frac{125}{3} - \frac{8}{3}\right) = 39 \ \ \textit{square unit}$



Example (10) : Find the area which is bounded by $x = 1 + y^2$ & the line $x = 10$?

Solution:

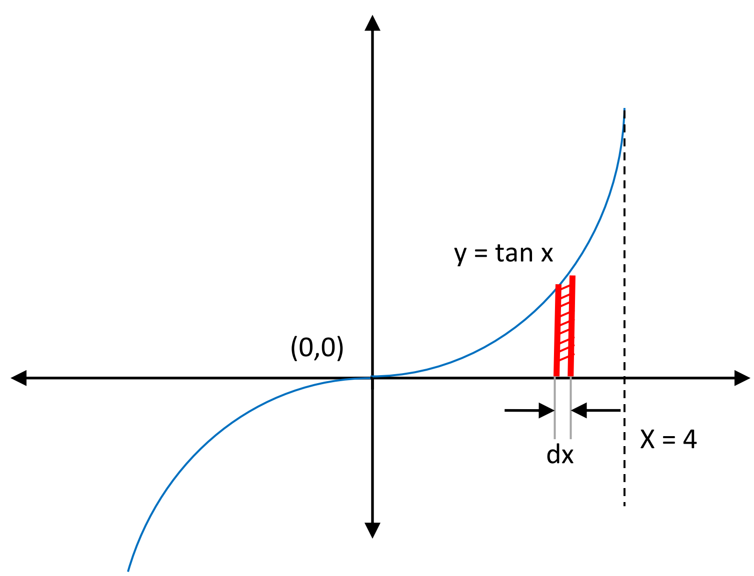
$A = \int_{1}^{10} \sqrt{x - 1} dx = \frac{2}{3} (x - 1)^{3/2} \Big|_{1}^{10} = \frac{2}{3} (9)^{3/2} - 0 = 36 \ \ \textit{square unit}$



Example (11) : Find the area which is bounded by y = tan x from x = 0 & x = π / 4 ?

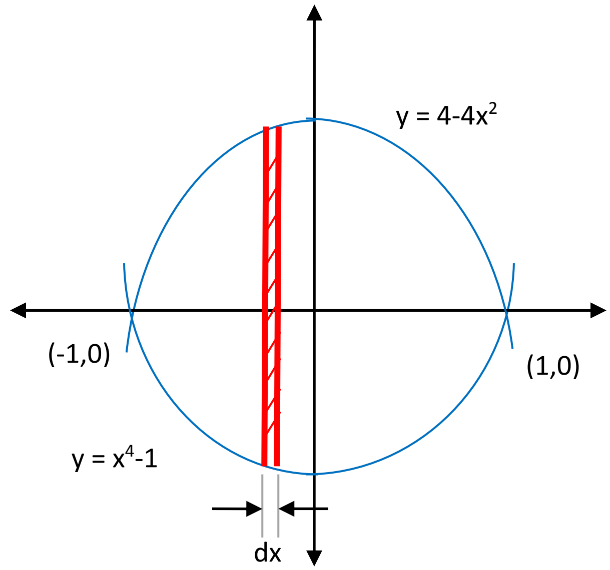
Solution:

$$A = \int_0^{\pi/4} \tan x \, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx = -\ln|\cos x|_0^{\pi/4} = 0.347 - 0 = 0.347 \quad square\ unit$$



Example (12) : **Find the area which is bounded by $y = 4 - 4x^2$ and $y = x^4 - 1$?**

Solution:



$x^4 - 1 = 4 - 4x^2$

$(x^2 + 5)(x^2 - 1) = 0 \Longrightarrow x^2 = -5$ (neglected)

$x^2 = 1 \Longrightarrow x = \pm 1$ & $y = 0$

$A = \int_{-1}^{1} (4 - 4x^2 - x^4 + 1)dx = 4x - \frac{4}{3}x^3 - \frac{x^5}{5} + x \Big|_{-1}^{1}$

$= \frac{104}{15}$ *square unit*

Chapter Eleven

METHODS OF INTEGRATION

11.1. Integration by Parts :

$$\int u\,dv = uv - \int v\,du$$

Example (1) : **Find $\int x \cos x\, dx$?**

Solution:

u = x **dv = cos x dx**

du = dx ⟵ **v = sin x**

= x sin x – $\int \sin x\, dx = x \sin x + \cos x + c$

Or :

u = cos x ↘ **dv = x dx**

du = - sin x dx ⟵ **v = $x^2/2$**

= $\frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x\, dx$ **(and then the integral become more complicated)**

Example (2) : **Find $\int x\, e^{-x}\, dx$?**

Solution:

u = x ↘ **dv = e^{-x} dx**

du = dx ⟵ **v = - e^{-x}**

= - x e^{-x} + $\int e^{-x} dx = -x\, e^{-x} - e^{-x} + c$

Example (3) : **Find $\int \ln x\, dx$?**

Solution:

u = ln x ↘ **dv = dx**

du = (1/x) dx ⟵ **v = x**

= x ln x - $\int dx = x \ln x - x + c$

Example (4) : Find $\int x^2\, e^x\, dx$?

Solution:

u = x^2 dv = e^x dx

du = 2x dx   ←   v = e^x

= $x^2 e^x - 2\int x\, e^x dx$

u = x   dv = e^x dx

du = dx   ←   v = e^x

= $x^2 e^x - 2\,[x\, e^x + \int e^x dx\,]$

= $x^2 e^x - 2x\, e^x + 2e^x + c$

Example (5) : Find $\int e^x \cos x\, dx$?

Solution:

u = e^x dv = cos x dx

du = e^x dx   ←   v = sin x

= $e^x \sin x - \int e^x \sin x\, dx$

u = e^x dv = sin x dx

du = e^x dx   ←   v = - cos x

= $e^x \sin x - [\, -e^x \cos x + \int e^x \cos x\, dx\,]$

$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + c$

$\int e^x \cos x\, dx = \frac{1}{2}(\, e^x \sin x + e^x \cos x\,) + c$

Example (6) : Find $\int x \sec^2 x \, dx$?

Solution:

$u = x$ $\quad\quad$ $dv = \sec^2 x \, dx$

$du = dx$ $\quad\quad$ $v = \tan x$

$= x \tan x - \int \tan x \, dx = x \tan x + \ln|\cos x| + c$

Example (7) : Find $\int (r^2 + r + 1)e^r \, dr$?

Solution:

$u = r^2 + r + 1$ $\quad\quad$ $dv = e^r \, dr$

$du = (2r + 1) \, dr$ $\quad\quad$ $v = e^r$

$= (r^2 + r + 1) e^r - \int (2r + 1) \, e^r dr$

$u = (2r + 1)$ $\quad\quad$ $dv = e^r \, dr$

$du = 2 \, dr$ $\quad\quad$ $v = e^r$

$= (r^2 + r + 1) e^r - (2r + 1)e^r + 2 \int e^r dr$

$= (r^2 + r + 1) e^r - (2r + 1)e^r + 2e^r + c$

Example (8) : Find $\int \tan^{-1} y \, dy$?

Solution:

$u = \tan^{-1} y$ $\qquad$ $dv = dy$

$du = \frac{1}{1+y^2} \, dy$ $\qquad$ $v = y$

$= y \tan^{-1} y - \int \frac{y}{1+y^2} \, dy = y \tan^{-1} y - \frac{1}{2} \int \frac{du}{u}$

$= y \tan^{-1} y - 0.5 \ln (1 + y^2) + c$

Example (9) : Find $\int t \sec^{-1} t \, dt$?

Solution:

$u = \sec^{-1} t$ $\qquad$ $dv = t \, dt$

$du = \frac{dt}{t\sqrt{t^2-1}}$ $\qquad$ $v = t^2 / 2$

$= 0.5 \, t^2 \sec^{-1} t - \frac{1}{2} \int \frac{t \, dt}{\sqrt{t^2-1}}$

$u = t^2 - 1 \quad \& \quad \frac{1}{4} \, du = \frac{1}{2} \, t \, dt$

$= 0.5 \, t^2 \sec^{-1} t - \frac{1}{4} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \, t^2 \sec^{-1} t - \frac{1}{4} * \frac{u^{1/2}}{1/2} + c$

$$= \frac{1}{2} \, t^2 \sec^{-1} t - \frac{1}{2} \sqrt{t^2 - 1} + c$$

Example (10) : Find $\int x^3 \cos 2x \, dx$?

Solution:

$u = x^3$ → $dv = \cos 2x\,dx$

$du = 3x^2\,dx$ ← $v = \frac{1}{2}\sin 2x$

$= 0.5\,x^3 \sin 2x \ - \ \frac{3}{2}\int x^3 \sin 2x\,dx$

$u = x^2$ → $dv = \sin 2x\,dx$

$du = 2x\,dx$ ← $v = \frac{1}{2}\cos 2x$

$= 0.5\,x^3 \sin 2x \ - \ \frac{3}{2}[-\frac{1}{2}\,x^2\cos 2x + \int x\cos 2x\ dx\,] = \frac{1}{2}\,x^3\sin 2x + \frac{3}{4}\,x^2\cos 2x - \frac{3}{2}\int x\cos 2x\,dx$

$u = x$ → $dv = \cos 2x\,dx$

$du = dx$ ← $v = \frac{1}{2}\sin 2x$

$= 0.5\,x^3 \sin 2x \ + \ \frac{3}{4}\,x^2\cos 2x - \frac{3}{4}\,x\sin 2x - \frac{3}{8}\cos 2x + c$

11.2. Integration by Tabular Method :

Example (1) : Find $\int x^2 e^x\,dx$?

Solution:

f (x) D	g (x) I
x^2	e^x
$2x$	e^x
2	e^x
0	e^x

$$\int x^2 e^x\,dx = x^2 e^x - 2x\,e^x + 2\,e^x + c$$

Example (2) : Find $\int x^3 \sin x\,dx$?

Solution:

f (x) D	g (x) I
x^3	Sin x
$3x^2$	- cos x
6 x	- sin x
6	Cos x
0	Sin x

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$$

Example (3) : **Solve the differential equation** $\frac{dy}{dx} = x^2 e^{4x}$

Solution:

$\int dy = \int x^2 e^{4x}\, dx$

$y = \int x^2 e^{4x}\, dx$

f (x) D	g (x) I
x^2	e^{4x}
2 x	$\frac{1}{4} e^{4x}$
2	$\frac{1}{16} e^{4x}$
0	$\frac{1}{64} e^{4x}$

$$= \frac{1}{4} x^2 e^{4x} - \frac{1}{8} x e^{4x} + \frac{1}{32} e^{4x} + c$$

Home Work :

$$\frac{dy}{dx} = x^2 \ln x$$

Example (4) : find the area of the region enclosed by y = x cos x and x – axis ?

Solution:

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx$$

f (x) D	g (x) I
x	cos x
1	$\sin x$
0	$-\cos x$

$= x \sin x + \cos x \,\Big|_0^{\pi/2} = \left[\frac{\pi}{2} * 1 + 0\right] - 1 = \frac{\pi - 1}{2}$

11.3. Integration by Partial Fraction Method :

Example (1) : Find $\int \frac{5x-3}{x^2-2x-3}\, dx$?

Solution:

$$\int \frac{5x-3}{x^2-2x-3}\, dx = \frac{A}{x+1} + \frac{B}{x-3}$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} \Rightarrow \frac{5x-3}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

$$\frac{(x+1)(x-3)(5x-3)}{(x^2-2x-3)} = A(x-3) + B(x+1)$$

5x – 3 = A (x – 3) + B (x + 1)

5 x – 3 = Ax – 3A + Bx + B ⟹ 5 x – 3 = (A + B) x – 3A + B

A + B = 5 ….. (1)

-3 A + B = - 3 …. (2)

-3 (5 – B) + B = -3 ⟹ - 15 + 3 B + B = -3 ⟹ B = 3 & A = 2

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1}dx + \int \frac{3}{x-3}\,dx$$

$$= 2\,ln\,|x+1| + 3\,ln\,|x-3| + c$$

Example (2) : Find $\int \frac{6x+7}{(x+2)^2}\,dx$?

Solution:

$$\int \frac{6x+7}{(x+2)^2}\,dx = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2} \Rightarrow 6x+7 = Ax+2A+B$$

A = 6 & 2 A + B = 7 ⟹ B = - 5

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \frac{6}{x+2}dx - \int \frac{5}{(x+2)^2}\,dx$$

$$= 6\,ln\,|x+2| - \frac{5}{(x+2)} + c$$

11.4. Integration by Heaviside (Cover - up) Method :

Example (1) : Find $\int \frac{x+4}{x^3+3x^2-10\,x}\,dx$?

Solution:

$\frac{x+4}{x(x-2)(x+5)}$, $x_1 = 0$, $x_2 = 2$, $x_3 = -5$

$$A_1 = \frac{0+4}{x\,(0-2)(0+5)} = \frac{-2}{5}$$

$$A_2 = \frac{2+4}{2(x-2)(2+5)} = \frac{3}{7}$$

$$A_3 = \frac{-5+4}{-5(-5-2)(x+5)} = \frac{-1}{35}$$

Therefore :

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)}$$

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = \frac{-2}{5} ln|x| + \frac{3}{7} ln|x-2| - \frac{1}{35} ln\,|x+5| + c$$

تمت بعونه تعالى